QUALITATIVE BEHAVIOR OF SOLUTIONS FOR THERMODYNAMICALLY CONSISTENT STEFAN PROBLEMS WITH SURFACE TENSION

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ABSTRACT. The qualitative behavior of a thermodynamically consistent two-phase Stefan problem with surface tension and with or without kinetic undercooling is studied. It is shown that these problems generate local semiflows in well-defined state manifolds. If a solution does not exhibit singularities in a sense made precise below, it is proved that it exists globally in time and its orbit is relatively compact. In addition, stability and instability of equilibria is studied. In particular, it is shown that multiple spheres of the same radius are unstable, reminiscent of the onset of Ostwald ripening.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^2, $n \geq 2$. Ω is occupied by a material that can undergo phase changes: at time t, phase i occupies the subdomain $\Omega_i(t)$ of Ω, respectively, with $i = 1, 2$. We assume that $\partial\Omega_1(t) \cap \partial\Omega = \emptyset$; this means that no *boundary contact* can occur. The closed compact hypersurface $\Gamma(t) := \partial\Omega_1(t) \subset \Omega$ forms the interface between the phases. By the *Stefan problem with surface tension* we mean the following problem.

Find a family of closed compact hypersurfaces $\{\Gamma(t)\}_{t\geq 0}$ contained in Ω and an appropriately smooth function $u : \mathbb{R}_+ \times \bar{\Omega} \to \mathbb{R}$ such that

$$\left\{\begin{aligned}
\kappa(u)\partial_t u - \operatorname{div}(d(u)\nabla u) &= 0 && \text{in} \quad \Omega \setminus \Gamma(t)\\
\partial_{\nu_\Omega} u &= 0 && \text{on} \quad \partial\Omega\\
[\![u]\!] &= 0 && \text{on} \quad \Gamma(t)\\
[\![\psi(u)]\!] + \sigma\mathcal{H} &= \gamma(u)V && \text{on} \quad \Gamma(t)\\
[\![d(u)\partial_\nu u]\!] &= (l(u) - \gamma(u)V)V && \text{on} \quad \Gamma(t)\\
u(0) &= u_0 &&\\
\Gamma(0) &= \Gamma_0. &&
\end{aligned}\right. \tag{1.1}$$

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Here $u(t)$ denotes the (absolute) temperature field, $\nu(t)$ the outer normal field of $\Omega_1(t)$, $V(t)$ the normal velocity of $\Gamma(t)$, $\mathcal{H}(t) = \mathcal{H}(\Gamma(t)) = -\mathrm{div}_{\Gamma(t)}\nu(t)/(n-1)$ the mean curvature of $\Gamma(t)$, and $[\![v]\!] = v_2|_{\Gamma(t)} - v_1|_{\Gamma(t)}$ the jump of a quantity v across $\Gamma(t)$. The sign of the mean curvature $\mathcal{H}$ is chosen to be negative at a point $x \in \Gamma$ if $\Omega_1 \cap B_r(x)$ is convex for some sufficiently small $r > 0$. Thus if Ω_1 is a ball of radius R then $\mathcal{H} = -1/R$ for its boundary Γ.

Several quantities are derived from the free energies $\psi_i(u)$ as follows:

- $\epsilon_i(u) := \psi_i(u) + u\eta_i(u)$, the internal energy in phase i,
- $\eta_i(u) := -\psi_i'(u)$, the entropy,
- $\kappa_i(u) := \epsilon_i'(u) = -u\psi_i''(u) > 0$, the heat capacity,
- $l(u) := u[\![\psi'(u)]\!] = -u[\![\eta(u)]\!]$, the latent heat.

Furthermore, $d_i(u) > 0$ denotes the coefficient of heat conduction in Fourier's law, $\gamma(u) \geq 0$ the coefficient of kinetic undercooling, and $\sigma > 0$ the coefficient of surface tension. In the sequel we drop the index i, as there is no danger of confusion; we just keep in mind that the coefficients depend on the phases.

The temperature is assumed to be continuous across the interface. However, the free energy and the conductivities depend on the respective phases, and hence the jumps $[\![\psi(u)]\!]$, $[\![\kappa(u)]\!]$, $[\![\eta(u)]\!]$, $[\![d(u)]\!]$ are in general non-zero at the interface. In this paper we assume that the coefficient of surface tension is constant. Then this model is consistent with the laws of thermodynamics. In fact, the total energy of the system is given by

$$\mathsf{E}(u,\Gamma) = \int_{\Omega\setminus\Gamma} \epsilon(u)dx + \frac{1}{n-1}\int_\Gamma \sigma\, ds, \tag{1.2}$$

and by the transport and surface transport theorem we have for smooth solutions

$$\begin{aligned}\frac{d}{dt}\mathsf{E}(u(t),\Gamma(t)) &= -\int_\Gamma \{[\![d(u)\partial_\nu u]\!] + [\![\epsilon(u)]\!]V + \sigma\mathcal{H}V\}\, ds \\ &= -\int_\Gamma \{[\![d(u)\partial_\nu u]\!] - (l(u) - \gamma(u)V))V\}\, ds = 0,\end{aligned}$$

and thus, energy is conserved. Let us point out that it is essential that $\sigma > 0$ is constant, i.e. is independent of temperature. The reason for this lies in the fact that in case $\sigma = \sigma(u)$ depends on the temperature, the surface energy will be $\int_\Gamma \epsilon_\Gamma(u)\, ds$ instead of $\int_\Gamma \sigma\, ds$, where $\epsilon_\Gamma(u) = \sigma(u) + u\eta_\Gamma(u)$, $\eta_\Gamma(u) = -\sigma'(u)$, and one has to take into account the surface entropy $\int_\Gamma \eta_\Gamma\, ds$ as well as balance of surface energy. This means that the Stefan law needs to be replaced by a dynamic boundary condition of the form

$$\kappa_\Gamma(u)\partial_{t,n}u - \mathrm{div}_\Gamma(d_\Gamma(u)\nabla_\Gamma u) = [\![d\partial_\nu u]\!] - \big(l(u) - \gamma(u)V + l_\Gamma(u)\mathcal{H}\big)V,$$

where $\partial_{t,n}$ denotes the time derivative in normal direction, $\kappa_\Gamma(u) = \epsilon_\Gamma'(u)$ and $l_\Gamma(u) = u\sigma'(u)$. We intend to study such more complex problems elsewhere and we restrict our attention to the case of constant σ, here.

The fifth equation in (1.1) is usually called the *Stefan law*. It shows that energy is conserved across the interface. The fourth equation is the famous *Gibbs-Thomson*

law (with kinetic undercooling if $\gamma(u) > 0$) which implies together with Stefan's law that entropy production on the interface is nonnegative if $\gamma \geq 0$. In case $\gamma \equiv 0$, i.e. in the absence of kinetic undercooling, there is no entropy production on the interface, see below. Indeed, the total entropy of the system, given by

$$\Phi(u, \Gamma) = \int_{\Omega\setminus\Gamma} \eta(u)dx, \tag{1.3}$$

satisfies

$$\begin{aligned}\frac{d}{dt}\Phi(u(t), \Gamma(t)) &= \int_\Omega \frac{1}{u^2} d(u)|\nabla u|^2\, dx - \int_\Gamma \frac{1}{u}\{[\![d(u)\partial_\nu u]\!] + u[\![\eta(u)]\!]V\}\, ds \\ &= \int_\Omega \frac{1}{u^2} d(u)|\nabla u|^2\, dx + \int_\Gamma \frac{1}{u}\gamma(u)V^2\, ds \geq 0.\end{aligned}$$

In particular, the negative total entropy is a Lyapunov functional for problem (1.1). Even more, $-\Phi$ is a strict Lyapunov functional in the sense that it is strictly decreasing along smooth solutions which are non-constant in time. Indeed, if at some time $t_0 \geq 0$ we have

$$\frac{d}{dt}\Phi(u(t_0), \Gamma(t_0)) = \int_\Omega \frac{1}{u^2} d(u)|\nabla u|^2\, dx + \int_\Gamma \frac{1}{u}\gamma(u)V^2\, ds = 0,$$

then $\nabla u(t_0) = 0$ in Ω and $\gamma(u(t_0))V(t_0) = 0$ on $\Gamma(t_0)$. This implies $u(t_0) = const$ in Ω, hence $\mathcal{H}(t_0) = -[\![\psi(u(t_0))]\!]/\sigma = const$. Since Ω is bounded, we may conclude that $\Gamma(t_0)$ is a union of finitely many, say m, disjoint spheres of equal radius, i.e. $(u(t_0), \Gamma(t_0))$ is an equilibrium. Therefore, the *limit sets* of solutions in the state manifold $\mathcal{SM}_\gamma$, see (3.8)-(3.9) for a definition, are contained in the $(mn + 1)$ dimensional manifold of equilibria

$$\begin{aligned}\mathcal{E} = \big\{\big(u_*, \bigcup_{1\leq l\leq m} S_{R_*}(x_l)\big) : u_* > 0,\ R_* = \sigma/[\![\psi(u_*)]\!],\ \bar{B}_{R_*}(x_l) \subset \Omega, \\ \text{and } S_{R_*}(x_l) \cap S_{R_*}(x_k) = \emptyset,\ l \neq k\big\},\end{aligned} \tag{1.4}$$

where $S_{R_*}(x_l)$ denotes the sphere with radius R_* and center x_l.

Another interesting observation is the following. Consider the critical points of the functional $\Phi(u, \Gamma)$ with constraint $\mathsf{E}(u, \Gamma) = \mathsf{E}_0$, say on $C(\bar{\Omega}) \times \mathcal{MH}^2(\Omega)$, see Section 3.1 for the definition of $\mathcal{MH}^2(\Omega)$. Then by the method of Lagrange multipliers, there is $\mu \in \mathbb{R}$ such that at a critical point (u_*, Γ_*) we have

$$\Phi'(u_*, \Gamma_*) + \mu\mathsf{E}'(u_*, \Gamma_*) = 0. \tag{1.5}$$

The derivatives of the functionals are given by

$$\langle\Phi'(u, \Gamma)|(v, h)\rangle = (\eta'(u)|v)_{L_2(\Omega)} - ([\![\eta(u)]\!]|h)_{L_2(\Gamma)},$$

and

$$\langle\mathsf{E}'(u, \Gamma)|(v, h)\rangle = (\epsilon'(u)|v)_{L_2(\Omega)} - ([\![\epsilon(u)]\!] + \sigma\mathcal{H}(\Gamma)|h)_{L_2(\Gamma)}.$$

Setting first $h=0$ and varying v in (1.5) we obtain $\eta'(u_*)+\mu\epsilon'(u_*)=0$ in Ω, and then varying h we get

$$[\![\eta(u_*)]\!]+\mu\big([\![\epsilon(u_*)]\!]+\sigma\mathcal{H}(\Gamma_*)\big)=0 \text{ on } \Gamma_*.$$

The relations $\eta(u)=-\psi'(u)$ and $\epsilon(u)=\psi(u)-u\psi'(u)$ imply $0=-\psi''(u_*)(1+\mu u_*)$, and this shows that $u_*=-1/\mu$ is constant in Ω, since $\kappa(u)=-u\psi''(u)>0$ for all $u>0$ by assumption. This further implies $[\![\psi(u_*)]\!]+\sigma\mathcal{H}(\Gamma_*)=0$, i.e. the Gibbs-Thomson relation. Since u_* is constant we see that $\mathcal{H}(\Gamma_*)$ is constant, hence Γ_* is a sphere whenever connected, and a union of finitely many disjoint spheres of equal size otherwise. Thus the critical points of the entropy functional for prescribed energy are precisely the equilibria of the problem.

Going further, suppose we have an equilibrium $e_*:=(u_*,\Gamma_*)$ where the total entropy has a local maximum, w.r.t. the constraint $\mathsf{E}=\mathsf{E}_0$ constant. Then $\mathcal{D}:=[\Phi+\mu\mathsf{E}]''(e_*)$ is negative semi-definite on the kernel of $\mathsf{E}'(e_*)$, where μ is the fixed Lagrange multiplier found above. The kernel of $\mathsf{E}'(e)$ is given by the identity

$$\int_\Omega \kappa(u)v\,dx-\int_\Gamma\big([\![\epsilon(u)]\!]+\sigma\mathcal{H}(\Gamma)\big)h\,ds=0,$$

which at equilibrium yields

$$\int_\Omega \kappa_* v\,dx+\int_{\Gamma_*} l_* h\,ds=0, \tag{1.6}$$

where $\kappa_*:=\kappa(u_*)$ and $l_*:=l(u_*)$. On the other hand, a straightforward calculation yields with $z=(v,h)$

$$-\langle\mathcal{D}z|z\rangle=\frac{1}{u_*}\left[\frac{1}{u_*}\int_\Omega \kappa_* v^2\,dx-\sigma\int_{\Gamma_*} h\cdot\mathcal{H}'(\Gamma_*)h\,ds\right]. \tag{1.7}$$

As κ_* and σ are positive, we see that the form $\langle\mathcal{D}z|z\rangle$ is negative semi-definite as soon as $\mathcal{H}'(\Gamma_*)$ is negative semi-definite. We have

$$\mathcal{H}'(\Gamma_*)=1/R_*^2+(1/(n-1))\Delta_{\Gamma_*},$$

where Δ_{Γ_*} denotes the Laplace-Beltrami operator on Γ_* and R_* means the radius of the equilibrium sphere. To derive necessary conditions for an equilibrium e_* to be a local maximum of entropy, we consider two cases.

1. Suppose that Γ_* is not connected, i.e. Γ_* is a union of m disjoint spheres Γ_*^k. Set $v=0$, and let $h=h_k\neq 0$ be constant on Γ_*^k (and zero on the remaining components of Γ_*) with $\sum_k h_k=0$. Then the constraint (1.6) holds, and

$$\langle\mathcal{D}z|z\rangle=(\sigma/u_*R_*^2)(|\Gamma_*|/m)\sum_k h_k^2>0,$$

hence $\mathcal{D}$ cannot be negative semi-definite in this case. Thus if e_* is an equilibrium with maximal total entropy, then Γ_* must be connected, and hence both phases are connected.

2. Assume that Γ_* is connected. Setting $v = l_*/(\kappa_*|1)_\Omega$ and $h = -1/|\Gamma_*|$, we see that $\mathcal{D}$ negative semi-definite on the kernel of $\mathsf{E}'(e_*)$ implies the condition

$$\zeta_* := \frac{\sigma u_*(\kappa_*|1)_\Omega}{l_*^2 R_*^2 |\Gamma_*|} \leq 1. \tag{1.8}$$

This is exactly the stability condition found in Theorem 4.5.

In summary, we have shown that:

- The equilibria of (1.1) are precisely the critical points of the entropy functional with prescribed energy.
- The entropy functional with prescribed energy does not have local maxima in case Γ_* is not connected.
- In case Γ_* is connected, a necessary condition for a critical point (u_*, Γ_*) to be a local maximum of the entropy functional with prescribed energy is inequality (1.8).

It will be shown in Theorems 4.5 and 5.2 below that

- $(u_*, \Gamma_*) \in \mathcal{E}$ is stable if Γ_* is connected and $\zeta_* < 1$.
- The latter is exactly the case if the reduced energy functional,
$$[u \mapsto \varphi(u) = \mathsf{E}(u, S_{R(u)}(x_0))], \quad R(u) = \sigma/[\![\psi(u)]\!],$$
has a strictly negative derivative at u_*.
- Any solution starting in a neighborhood of a stable equilibrium exists globally and converges to another stable equilibrium exponentially fast.
- $(u_*, \Gamma_*) \in \mathcal{E}$ is always unstable if Γ_* is disconnected, or if $\zeta_* > 1$.

Hence multiple spheres (of the same radius) are always unstable for (1.1). This situation is reminiscent of the onset of *Ostwald ripening*.

The authors in [1] also consider the case with the Gibbs-Thomson law replaced by

$$[\![\psi(u)]\!] + \sigma\mathcal{H} = \gamma(u)V - \mathrm{div}_\Gamma[\alpha(u)\nabla_\Gamma(V/u)] - u\,\mathrm{div}_\Gamma[\beta(u)\nabla_\Gamma V],$$

where $\alpha, \beta > 0$ may also depend on the temperature. The modified Stefan law in this case reads

$$[\![d(u)\partial_\nu u]\!] = \Big(l(u) - \gamma(u)V + \mathrm{div}_\Gamma[\alpha(u)\nabla_\Gamma(V/u)] + u\,\mathrm{div}_\Gamma[\beta(u)\nabla_\Gamma V]\Big)V.$$

The resulting entropy production then becomes

$$\begin{aligned}\frac{d}{dt}\Phi(u(t),\Gamma(t)) &= \int_\Omega \frac{1}{u^2} d(u)|\nabla u|^2\,dx + \int_\Gamma \frac{1}{u}\gamma(u)V^2\,ds \\ &\quad + \int_\Gamma [\alpha(u)|\nabla_\Gamma(V/u)|^2 + \beta(u)|\nabla_\Gamma V|^2]|\,ds \geq 0.\end{aligned}$$

Although interesting from an analytic point of view, these models will not be considered here.

Now we want to relate problem (1.1) to the Stefan problems that have been studied in the mathematical literature so far. For this purpose we linearize $h(u) := [\![\psi(u)]\!]$ near the *melting temperature* u_m defined by $h(u_m) = 0$. Then for the relative temperature $v = u - u_m$ we have $h(u) \approx h'(u_m)v$, hence with $l_m = l(u_m)$ and $\gamma_m = \gamma(u_m)$, the Gibbs-Thomson law approximately becomes

$$(l_m/u_m)v + \sigma\mathcal{H} = \gamma_m V. \tag{1.9}$$

This is the classical Gibbs-Thomson law with kinetic undercooling. Similarly, assuming that u is close to u_m and V is small, the Stefan law becomes approximately

$$[\![d\partial_\nu u]\!] = l_m V. \tag{1.10}$$

In the literature it is frequently assumed that the heat capacities are constant and equal in the phases. Then we necessarily have

$$0 \equiv [\![\kappa]\!] = [\![\epsilon'(u)]\!] = -u[\![\psi''(u)]\!],$$

which implies that the function $h(u) = [\![\psi(u)]\!]$ is linear, i.e. $h(u) = h_0 + h_1 u$, and then $l(u) = h_1 u$. The melting temperature is here given by $0 < u_m = -h_0/h_1$.

If the heat capacities κ_i are constant in the phases but not necessarily equal, the internal energies depend linearly on the temperature and the free energies are of the form $\psi_i(u) = a_i + b_i u - \kappa_i u \ln u$, hence $h(u) = \alpha + \beta u - \delta u \ln(u)$, with constants $\alpha, \beta, \delta \in \mathbb{R}$. Concerning existence of equilibria, these special cases will be discussed in more detail in Section 4.

Let us now briefly discuss the literature concerning problem (1.1) and related problems. We refer to [1] and [22] for information on the modeling aspects of (1.1), see also Gurtin [18, 19] for earlier work. Additional information for the classical Gibbs-Thomson law (1.9) can be found in [3, 4, 18, 19, 21, 24, 38].

Existence and regularity results for the Stefan problem with the classical Gibbs-Thomson law $u = \sigma\mathcal{H}$ and the classical Stefan law (1.10) in case $\kappa_1 = \kappa_2$ can be found in [14, 15, 16, 26, 27, 35, 36], see [14] for a detailed discussion. The Stefan problem with the Gibbs-Thomson law (1.9) and the classical Stefan law (1.10) in case $\kappa_1 = \kappa_2$ has been studied in [7, 35, 36, 37], see also [23] for the one-phase case.

If $\kappa_1 = \kappa_2 = 0$ then we obtain a thermodynamically consistent quasi-stationary approximation of the Stefan problem with surface tension (and kinetic undercooling). Existence, uniqueness, regularity, and global existence of solutions for the quasi-stationary approximation with the Gibbs-Thomson law $u = \sigma\mathcal{H}$ and the classical Stefan law (1.10) has been studied in [2, 5, 6, 10, 11, 12, 13, 17]. Existence and global existence of classical solutions for the quasi-stationary approximation with (1.9) and the classical Stefan law (1.10) has been investigated in [39, 23].

We refer to [30] for a detailed discussion of the literature of the classical Stefan problem with $\kappa_1 = \kappa_2$, $\sigma = \gamma = 0$ and (1.10).

It appears that this manuscript is the first work to provide analytical results for the thermodynamically consistent Stefan problem (1.1).

The plan for this paper is as follows. In Section 2 we first transform the problem to a domain with fixed interface, employing a *Hanzawa transform* [20]. This is the so-called *direct mapping approach*. Section 3 is devoted to results on local well-posedness for problem (1.1), based on the approach in [15] and [9]. We introduce the state manifolds $\mathcal{SM}_\gamma$ for (1.1) and we consider the local semiflow in $\mathcal{SM}_\gamma$ generated by problem (1.1). In Section 4 we discuss equilibria and their linear stability properties. Here we rely on previous work of the authors [32]. In Section 5 we establish the corresponding stability properties for the nonlinear problem, employing the *generalized principle of linearized stability*, extending the results of [33] to the situation considered here. The main result of this section shows convergence of solutions to an equilibrium which start out near stable equilibria. Moreover, we give a rigorous proof of the instability result. Of ultimate importance is the Lyapunov functional for (1.1), which is given by the negative total entropy $-\Phi(u,\Gamma)$. It takes bounded global-in-time solutions to the set of equilibria, and then by the results of Section 5 and compactness of the orbits, such solutions must converge towards an equilibrium in the topology of the state manifold $\mathcal{SM}_\gamma$ provided they come close to a stable equilibrium.

Our analysis is carried out in the framework of L_p-spaces, with $n+2<p<\infty$. We expect that it would be enough to require $(n+2)/2<p<\infty$ (so unfortunately $p>2$ even in $2D$!), but for the sake of simplicity we restrict ourselves here to the stronger assumption $p>n+2$. We also expect that a similar analysis can be obtained in the framework of the small Hölder spaces h^α, which would, though, require higher order compatibility conditions.

2. Transformation to a Fixed Interface

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^2, and suppose $\Gamma\subset\Omega$ is a closed hypersurface of class C^2, i.e. a C^2-manifold which is the boundary of a bounded domain $\Omega_1\subset\Omega$. We then set $\Omega_2=\Omega\setminus\bar\Omega_1$. Note that while Ω_2 is connected, Ω_1 may be disconnected. However, Ω_1 consists of finitely many components only, as $\partial\Omega_1=\Gamma$ by assumption is a manifold, at least of class C^2. Recall that the *second order bundle* of Γ is given by

$$\mathcal{N}^2\Gamma:=\{(p,\nu_\Gamma(p),L_\Gamma(p)):\,p\in\Gamma\}.$$

Note that the Weingarten map L_Γ (also called the shape operator, or the second fundamental tensor) is defined by

$$L_\Gamma(p)=-\nabla_\Gamma\nu_\Gamma(p),\quad p\in\Gamma,$$

where ∇_Γ denotes the surface gradient on Γ. The eigenvalues $\kappa_j(p)$ of $L_\Gamma(p)$ are the principal curvatures of Γ at $p\in\Gamma$, and we have $|L_\Gamma(p)|=\max_j|\kappa_j(p)|$. The mean curvature $\mathcal{H}_\Gamma(p)$ is given by

$$(n-1)\mathcal{H}_\Gamma(p)=\sum_{j=1}^{n-1}\kappa_j(p)=\mathrm{tr}L_\Gamma(p)=-\mathrm{div}_\Gamma\nu_\Gamma(p),$$

where div_Γ means surface divergence. Recall also that the *Hausdorff distance* d_H between the two closed subsets $A, B \subset \mathbb{R}^m$ is defined by

$$d_H(A,B) := \max\big\{\sup_{a\in A}\mathrm{dist}(a,B), \sup_{b\in B}\mathrm{dist}(b,A)\big\}.$$

Then we may approximate Γ by a real analytic hypersurface Σ (or merely $\Sigma \in C^3$), in the sense that the Hausdorff distance of the second order bundles of Γ and Σ is as small as we want. More precisely, for each $\eta > 0$ there is a real analytic closed hypersurface such that $d_H(\mathcal{N}^2\Sigma, \mathcal{N}^2\Gamma) \le \eta$. If $\eta > 0$ is small enough, then Σ bounds a domain Ω_1^Σ with $\overline{\Omega_1^\Sigma} \subset \Omega$, and we set $\Omega_2^\Sigma = \Omega \setminus \bar{\Omega}_1^\Sigma$.

It is well known that such a hypersurface Σ admits a tubular neighborhood, which means that there is $a > 0$ such that the map

$$\Lambda :\ \Sigma \times (-a,a) \to \mathbb{R}^n$$
$$\Lambda(p,r) := p + r\nu_\Sigma(p)$$

is a diffeomorphism from $\Sigma \times (-a,a)$ onto $\mathcal{R}(\Lambda)$. The inverse

$$\Lambda^{-1} : \mathcal{R}(\Lambda) \mapsto \Sigma \times (-a,a)$$

of this map is conveniently decomposed as

$$\Lambda^{-1}(x) = (\Pi(x), d_\Sigma(x)), \quad x \in \mathcal{R}(\Lambda).$$

Here $\Pi(x)$ means the nonlinear orthogonal projection of x to Σ and $d_\Sigma(x)$ the signed distance from x to Σ; so $|d_\Sigma(x)| = \mathrm{dist}(x,\Sigma)$ and $d_\Sigma(x) < 0$ iff $x \in \Omega_1^\Sigma$. In particular we have $\mathcal{R}(\Lambda) = \{x \in \mathbb{R}^n :\ \mathrm{dist}(x,\Sigma) < a\}$.

On the one hand, a is determined by the curvatures of Σ, i.e. we must have

$$0 < a < \min\big\{1/|\kappa_j(p)| : j = 1,\dots,n-1,\ p \in \Sigma\big\},$$

where $\kappa_j(p)$ mean the principal curvatures of Σ at $p \in \Sigma$. But on the other hand, a is also connected to the topology of Σ, which can be expressed as follows. Since Σ is a compact (smooth) manifold of dimension $n-1$ it satisfies a (interior and exterior) ball condition, which means that there is a radius $r_\Sigma > 0$ such that for each point $p \in \Sigma$ there are $x_j \in \Omega_j^\Sigma$, $j = 1,2$, such that $B_{r_\Sigma}(x_j) \subset \Omega_j^\Sigma$, and $\bar{B}_{r_\Sigma}(x_j) \cap \Sigma = \{p\}$. Choosing r_Σ maximal, we then must also have $a < r_\Sigma$. In the sequel we fix

$$a = \frac{1}{2}\min\left\{r_\Sigma, \frac{1}{|\kappa_j(p)|}, j = 1,\dots,n-1,\ p \in \Sigma\right\}.$$

For later use we note that the derivatives of $\Pi(x)$ and $d_\Sigma(x)$ are given by

$$\nabla d_\Sigma(x) = \nu_\Sigma(\Pi(x)), \quad \Pi'(x) = M_0(d_\Sigma(x), \Pi(x))P_\Sigma(\Pi(x)),$$

where $P_\Sigma(p) = I - \nu_\Sigma(p) \otimes \nu_\Sigma(p)$ denotes the orthogonal projection onto the tangent space $T_p\Sigma$ of Σ at $p \in \Sigma$, and $M_0(r,p) = (I - rL_\Sigma(p))^{-1}$. Note that $|M_0(r,p)| \le 1/(1 - r|L_\Sigma(p)|) \le 2$ for all $|r| \le a$ and $p \in \Sigma$.

Setting $\Gamma = \Gamma(t)$, we may use the map Λ to parameterize the unknown free boundary $\Gamma(t)$ over Σ by means of a height function $\rho(t,p)$ via

$$\Gamma(t) : [p \mapsto p + \rho(t,p)\nu_\Sigma(p)], \quad p \in \Sigma,\ t \geq 0,$$

for small $t \geq 0$, at least. Extend this diffeomorphism to all of $\bar{\Omega}$ by means of

$$\Xi_\rho(t,x) = x + \chi(d_\Sigma(x)/a)\rho(t,\Pi(x))\nu_\Sigma(\Pi(x)) =: x + \theta_\rho(t,x).$$

Here χ denotes a suitable cut-off function; more precisely, $\chi \in \mathcal{D}(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(r) = 1$ for $|r| < 1/3$, and $\chi(r) = 0$ for $|r| > 2/3$. Note that $\Xi_\rho(t,x) = x$ for $|d_\Sigma(x)| > 2a/3$, and

$$\Xi_\rho^{-1}(t,x) = x - \rho(t,\Pi(x))\nu_\Sigma(\Pi(x)) \quad \text{for } \ |d_\Sigma(x)| < a/3.$$

In particular,

$$\Xi_\rho^{-1}(t,x) = x - \rho(t,x)\nu_\Sigma(x) \quad \text{for } \ x \in \Sigma.$$

Setting $v(t,x) = u(t,\Xi_\rho(t,x))$, or $u(t,x) = v(t,\Xi_\rho^{-1}(t,x))$ we have this way transformed the time varying regions $\Omega \setminus \Gamma(t)$ to the fixed domain $\Omega \setminus \Sigma$. This is the direct mapping method, also called Hanzawa transformation.

By means of this transformation, we obtain the following transformed problem.

$$\left\{\begin{aligned} \kappa(v)\partial_t v + \mathcal{A}(v,\rho)v &= \kappa(v)\mathcal{R}(\rho)v && \text{in} && \Omega \setminus \Sigma \\ \partial_{\nu_\Omega} v &= 0 && \text{on} && \partial\Omega \\ [\![v]\!] &= 0 && \text{on} && \Sigma \\ [\![\psi(v)]\!] + \sigma\mathcal{H}(\rho) &= \gamma(v)\beta(\rho)\partial_t\rho && \text{on} && \Sigma \\ \{l(v) - \gamma(v)\beta(\rho)\partial_t\rho\}\beta(\rho)\partial_t\rho + \mathcal{B}(v,\rho)v &= 0 && \text{on} && \Sigma \\ v(0) = v_0,\ \rho(0) &= \rho_0. \end{aligned}\right. \tag{2.1}$$

Here $\mathcal{A}(v,\rho)$ and $\mathcal{B}(v,\rho)$ denote the transformations of $-\mathrm{div}(d\nabla)$ and $-[\![d\partial_\nu]\!]$, respectively. Moreover, $\mathcal{H}(\rho)$ means the mean curvature of Γ, $\beta(\rho) = (\nu_\Sigma|\nu_\Gamma(\rho))$, the term $\beta(\rho)\partial_t\rho$ represents the normal velocity V, and

$$\mathcal{R}(\rho)v = \partial_t v - \partial_t u \circ \Xi_\rho.$$

The system (2.1) is a quasi-linear parabolic problem on the domain Ω with fixed interface $\Sigma \subset \Omega$ with a *dynamic boundary condition*, namely the fifth equation which describes the evolution of the interface $\Gamma(t)$.

To elaborate on the structure of this problem in more detail, we calculate

$$\Xi'_\rho = I + \theta'_\rho, \qquad \Xi'^{-1}_\rho = I - [I + \theta'_\rho]^{-1}\theta'_\rho =: I - M_1(\rho)^{\mathsf{T}}$$

and

$$\nabla u \circ \Xi_\rho = [(\Xi_\rho^{-1})'^{\mathsf{T}} \circ \Xi_\rho]\nabla v = (I - M_1(\rho))\nabla v,$$

and for a vector field $q = \bar{q} \circ \Xi_\rho$

$$(\nabla|\bar{q}) \circ \Xi_\rho = ([(\Xi_\rho^{-1})'^{\mathsf{T}} \circ \Xi_\rho]\nabla|q) = ((I - M_1(\rho))\nabla|q).$$

Further we have

$$\begin{aligned}\partial_t u\circ\Xi_\rho &= \partial_t v - (\nabla u\circ\Xi_\rho|\partial_t\Xi_\rho) = \partial_t v - ([(\Xi_\rho^{-1})'^{\mathsf T}\circ\Xi_\rho]\nabla v|\partial_t\Xi_\rho)\\ &= \partial_t v - (\nabla v|[I+\theta_\rho']^{-1}\partial_t\theta_\rho),\end{aligned}$$

hence

$$\mathcal{R}(\rho)v = (\nabla v|[I+\theta_\rho']^{-1}\partial_t\theta_\rho).$$

With the Weingarten map $L_\Sigma = -\nabla_\Sigma\nu_\Sigma$ we have

$$\begin{aligned}\nu_\Gamma(\rho) &= \beta(\rho)(\nu_\Sigma - \alpha(\rho)), \quad & \alpha(\rho) &= M_0(\rho)\nabla_\Sigma\rho,\\ M_0(\rho) &= (I-\rho L_\Sigma)^{-1}, & \beta(\rho) &= (1+|\alpha(\rho)|^2)^{-1/2},\end{aligned}$$

and

$$V = (\partial_t\Xi|\nu_\Gamma) = (\nu_\Sigma|\nu_\Gamma(\rho))\partial_t\rho = \beta(\rho)\partial_t\rho.$$

Employing this notation leads to $\theta_\rho' = 0$ for $|d_\Sigma(x)| > 2a/3$ and

$$\begin{aligned}\theta_\rho'(t,x) &= \frac{1}{a}\chi'(d_\Sigma(x)/a)\rho(t,\Pi(x))\nu_\Sigma(\Pi(x))\otimes\nu_\Sigma(\Pi(x))\\ &\quad+\chi(d_\Sigma(x)/a)[\nu_\Sigma(\Pi(x))\otimes M_0(d_\Sigma(x))\nabla_\Sigma\rho(t,\Pi(x))]\\ &\quad-\chi(d_\Sigma(x)/a)\rho(t,\Pi(x))L_\Sigma(\Pi(x))M_0(d_\Sigma(x))P_\Sigma(\Pi(x))\end{aligned}$$

for $0\le|d_\Sigma(x)|\le 2a/3$. In particular, for $x\in\Sigma$ we have

$$\theta_\rho'(t,x) = \nu_\Sigma(x)\otimes\nabla_\Sigma\rho(t,x) - \rho(t,x)L_\Sigma(x)P_\Sigma(x),$$

and

$$(\theta_\rho')^{\mathsf T}(t,x) = \nabla_\Sigma\rho(t,x)\otimes\nu_\Sigma(x) - \rho(t,x)L_\Sigma(x),$$

since $L_\Sigma(x)$ is symmetric and has range in $T_x\Sigma$. Therefore, $[I+\theta_\rho']$ is boundedly invertible, if ρ and $\nabla_\Sigma\rho$ are sufficiently small, and

$$|[I+\theta_\rho']^{-1}|\le 2 \quad \text{for} \;\; |\rho|_\infty \le \frac{1}{4(|\chi'|_\infty/a + 2\max_j|\kappa_j|)}, \quad |\nabla_\Sigma\rho|_\infty\le\frac{1}{8}.$$

For the mean curvature $\mathcal{H}(\rho)$ we have

$$(n-1)\mathcal{H}(\rho) = \beta(\rho)\{\mathrm{tr}[M_0(\rho)(L_\Sigma+\nabla_\Sigma\alpha(\rho))] - \beta^2(\rho)(M_0(\rho)\alpha(\rho)|[\nabla_\Sigma\alpha(\rho)]\alpha(\rho))\},$$

an expression involving second order derivatives of ρ only linearly. Its linearization at $\rho=0$ is given by

$$(n-1)\mathcal{H}'(0) = \mathrm{tr}\, L_\Sigma^2 + \Delta_\Sigma.$$

Here Δ_Σ denotes the Laplace-Beltrami operator on Σ. The operator $\mathcal{B}(v,\rho)$ becomes

$$\begin{aligned}\mathcal{B}(v,\rho)v &= -[\![d(u)\partial_\nu u]\!]\circ\Xi_\rho = -([\![d(v)(I-M_1(\rho))\nabla v]\!]|\nu_\Gamma)\\ &= -\beta(\rho)([\![d(v)(I-M_1(\rho))\nabla v]\!]|\nu_\Sigma-\alpha(\rho))\\ &= -\beta(\rho)[\![d(v)\partial_{\nu_\Sigma}v]\!] + \beta(\rho)([\![d(v)\nabla v]\!]|(I-M_1(\rho))^{\mathsf T}\alpha(\rho)),\end{aligned}$$

since $M_1^{\mathsf{T}}(\rho)\nu_\Sigma = 0$, and finally

$$\begin{aligned}\mathcal{A}(v,\rho)v &= -\operatorname{div}(d(u)\nabla u)\circ \Xi_\rho = -((I-M_1(\rho))\nabla|d(v)(I-M_1(\rho))\nabla v)\\ &= -d(v)\Delta v + d(v)[M_1(\rho)+M_1^{\mathsf{T}}(\rho)-M_1(\rho)M_1^{\mathsf{T}}(\rho)] : \nabla^2 v\\ &\quad - d'(v)|(I-M_1(\rho))\nabla v|^2 + d(v)((I-M_1(\rho)) : \nabla M_1(\rho)|\nabla v).\end{aligned}$$

We recall that for matrices $A, B \in \mathbb{R}^{n\times n}$, $A : B = \sum_{i,j=1}^n a_{ij}b_{ij} = \operatorname{tr}(AB^{\mathsf{T}})$ denotes the inner product.

Obviously, the leading part of $\mathcal{A}(v,\rho)v$ is $-d(v)\Delta v$, while the leading part of $\mathcal{B}(v,\rho)v$ is $-\beta(\rho)[\![d(v)\partial_{\nu_\Sigma} v]\!]$, as $M_1(0) = 0$ and $\alpha(0) = 0$; recall that we may assume ρ small in the C^2-norm. It is important to recognize the quasilinear structure of (2.1): derivatives of highest order only appear linearly in each of the equations.

3. Local Well-Posedness

The basic result for local well-posedness in the absence of kinetic undercooling in an L_p-setting is the following.

Theorem 3.1. ($\gamma \equiv 0$). *Let $p > n+2$, $\gamma = 0$, $\sigma > 0$. Suppose $\psi \in C^3(0,\infty)$, $d \in C^2(0,\infty)$ such that*

$$\kappa(u) = -u\psi''(u) > 0, \quad d(u) > 0, \quad u \in (0,\infty).$$

Assume the regularity conditions

$$u_0 \in W_p^{2-2/p}(\Omega\setminus\Gamma_0)\cap C(\bar\Omega), \quad u_0 > 0, \quad \Gamma_0 \in W_p^{4-3/p},$$

the compatibility conditions

$$[\![\psi(u_0)]\!] + \sigma\mathcal{H}(\Gamma_0) = 0, \quad [\![d(u_0)\partial_{\nu_{\Gamma_0}} u_0]\!] \in W_p^{2-6/p}(\Gamma_0),$$

and the well-posedness condition

$$l(u_0) \neq 0 \quad on\ \Gamma_0.$$

Then there exists a unique L_p-solution for the Stefan problem with surface tension (1.1) *on some possibly small but nontrivial time interval $J = [0,\tau]$.*

Here the notation $\Gamma_0 \in W_p^{4-3/p}$ means that Γ_0 is a C^2-manifold, such that its (outer) normal field ν_{Γ_0} is of class $W_p^{3-3/p}(\Gamma_0)$. Therefore the Weingarten tensor $L_{\Gamma_0} = -\nabla_{\Gamma_0}\nu_{\Gamma_0}$ of Γ_0 belongs to $W_p^{2-3/p}(\Gamma_0)$ which embeds into $C^{1+\alpha}(\Gamma_0)$, with $\alpha = 1-(n+2)/p > 0$ since $p > n+2$ by assumption. For the same reason, we also have $u_0 \in C^{1+\alpha}(\bar\Omega)$, and $V_0 \in C^{2\alpha}(\Gamma_0)$. The notion L_p-solution means that (u,Γ) is obtained as the push-forward of an L_p-solution (v,ρ) of the transformed problem (2.1). This class will be discussed below.

There is an analogous result in the presence of kinetic undercooling which reads as follows.

Theorem 3.2. ($\gamma > 0$). *Let $p > n+2$, $\sigma > 0$, and suppose $\psi, \gamma \in C^3(0,\infty)$, $d \in C^2(0,\infty)$ such that*

$$\kappa(u) = -u\psi''(u) > 0, \quad d(u) > 0, \quad \gamma(u) > 0, \quad u \in (0,\infty).$$

Assume the regularity conditions

$$u_0 \in W_p^{2-2/p}(\Omega \setminus \Gamma_0) \cap C(\bar\Omega), \quad u_0 > 0, \quad \Gamma_0 \in W_p^{4-3/p},$$

and the compatibility condition

$$\big([\![\psi(u_0)]\!] + \sigma\mathcal{H}(\Gamma_0)\big)\big(l(u_0) - [\![\psi(u_0)]\!] - \sigma\mathcal{H}(\Gamma_0)\big) = \gamma(u_0)[\![d(u_0)\partial_\nu u_0]\!].$$

Then there exists a unique L_p-solution of the Stefan problem with surface tension and kinetic undercooling (1.1) *on some possibly small but nontrivial time interval $J = [0,\tau]$.*

Proof of Theorems 3.1 and 3.2:
(i) *Direct mapping method: Hanzawa transformation.*
As explained in the previous section, we employ a Hanzawa transformation and study the resulting problem (2.1) on the domain Ω with fixed interface Σ.

In case $\gamma \equiv 0$, for the L_p-theory, the solution of the transformed problem will belong to the class

$$v \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega \setminus \Sigma)) \hookrightarrow C(J; W_p^{2-2/p}(\Omega \setminus \Sigma)),$$

$$\rho \in W_p^{1-1/2p}(J; H_p^2(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)) \hookrightarrow C(J; W_p^{4-3/p}(\Sigma)),$$

$$\partial_t\rho \in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-2/p}(\Sigma)) \hookrightarrow C(J; W_p^{2-6/p}(\Sigma)), \tag{3.1}$$

see [15] for a proof of the last two embeddings in the case $\Sigma = \mathbb{R}^n$.

If $\gamma > 0$ we have moreover

$$\rho \in W_p^{2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)) \hookrightarrow C^1(J; W_p^{2-3/p}(\Sigma)).$$

Note that in both cases, $v \in C(J \times \bar\Omega)$, $v|_{\Omega_j} \in C(J; C^1(\bar\Omega_j))$, $j = 1,2$. Moreover, $\rho \in C(J; C^3(\Sigma))$ and

$$\partial_t\rho \in C(J; C(\Sigma)) \text{ in case } \gamma = 0, \quad \partial_t\rho \in C(J; C^1(\Sigma)) \text{ in case } \gamma > 0.$$

We set

$$\begin{aligned}
\mathbb{E}_1(J) &:= \{v \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega \setminus \Sigma) : [\![v]\!] = 0,\ \partial_{\nu_\Omega} v = 0\},\\
\mathbb{E}_2(J) &:= W_p^{3/2-1/2p}(J; L_p(\Sigma)) \cap W_p^{1-1/2p}(J; H_p^2(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)),\ \gamma \equiv 0,\\
\mathbb{E}_2(J) &:= W_p^{2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)),\ \gamma > 0,\\
\mathbb{E}(J) &:= \mathbb{E}_1(J) \times \mathbb{E}_2(J),
\end{aligned}$$

i.e. $\mathbb{E}(J)$ denotes the solution space. Similarly, we define

$$\begin{aligned}
\mathbb{F}_1(J) &:= L_p(J; L_p(\Omega)),\\
\mathbb{F}_2(J) &:= W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)),\\
\mathbb{F}_3(J) &:= W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)),\\
\mathbb{F}(J) &:= \mathbb{F}_1(J) \times \mathbb{F}_2(J) \times \mathbb{F}_3(J),
\end{aligned}$$

i.e. $\mathbb{F}(J)$ means the space of data. A left subscript zero means vanishing time trace at $t = 0$, whenever it exists. So for example

$$_0\mathbb{E}_2(J) = \{\rho \in \mathbb{E}_2(J) :\ \rho(0) = \partial_t\rho(0) = 0\}$$

whenever $p > 3$. Employing the calculations in Section 2 and splitting into the principal linear part and a nonlinear part, we arrive at the following formulation of problem (2.1).

$$\left\{\begin{aligned}
\kappa_0(x)\partial_t v - d_0(x)\Delta v &= F(v,\rho) && \text{in} && \Omega \setminus \Sigma\\
\partial_{\nu_\Omega} v &= 0 && \text{on} && \partial\Omega\\
[\![v]\!] &= 0, && \text{on} && \Sigma\\
l_1(t,x)v + \sigma_0\Delta_\Sigma\rho - \gamma_1(t,x)\partial_t\rho &= G(v,\rho) && \text{on} && \Sigma\\
l_0(x)\partial_t\rho - [\![d_0(x)\partial_\nu v]\!] &= H(v,\rho) && \text{on} && \Sigma\\
v(0) = v_0,\ \rho(0) &= \rho_0.
\end{aligned}\right. \tag{3.2}$$

Here

$$\begin{aligned}
&\kappa_0(x) = \kappa(v_0(x)), \quad d_0(x) = d(v_0(x)), \quad l_0(x) = l(v_0(x)), \quad \sigma_0 = \frac{\sigma}{n-1},\\
&l_1(t,\cdot) = [\![\psi'(e^{\Delta_\Sigma t}v_{0\Sigma})]\!], \quad \gamma_1(t,\cdot) = \gamma(e^{\Delta_\Sigma t}v_{0\Sigma}),
\end{aligned}$$

where $v_{0\Sigma}$ means the restriction of v_0 to Σ. Note that $\kappa_0, d_0 \in W_p^{2-2/p}(\Omega \setminus \Sigma)$, hence these functions are in $C^1(\bar\Omega_j)$, $j = 1, 2$. Recall that d and κ may be different in different phases. Further we have $l_0 \in W_p^{2-3/p}(\Sigma)$ which implies $l_0 \in C^1(\Sigma)$. This is good enough for the space $\mathbb{F}_3(J)$, as C^1-functions are pointwise multipliers for $\mathbb{F}_3(J)$, but it is not good enough for $\mathbb{F}_2(J)$. For this reason, we need to define the extension $v_b := e^{\Delta_\Sigma t}v_{0\Sigma}$. This function as well as l_1 and γ_1 belong to $\mathbb{F}_2(J)$, hence are pointwise multipliers for this space, as $\mathbb{F}_2(J)$ and $\mathbb{F}_3(J)$ are Banach algebras w.r.t. pointwise multiplication, as $p > n + 2$.

The nonlinearities F, G, and H are defined as follows.

$$\begin{aligned}
F(v,\rho) &= (\kappa_0 - \kappa(v))\partial_t v + (d(v) - d_0)\Delta v + d(v)M_2(\rho) : \nabla^2 v\\
&\quad - d'(v)|(I - M_1(\rho))\nabla v|^2 + d(v)(M_3(\rho)|\nabla v) + \kappa(v)\mathcal{R}(\rho)v,\\
G(v,\rho) &= -([\![\psi(v)]\!] + \sigma\mathcal{H}(\rho)) + l_1 v + \sigma_0\Delta_\Sigma\rho + (\gamma(v)\beta(\rho) - \gamma_1)\partial_t\rho,\\
H(v,\rho) &= [\![(d(v) - d_0)\partial_\nu v]\!] + (l_0 - l(v))\partial_t\rho + ([\![d(v)\nabla v]\!]|M_4(\rho)\nabla_\Sigma\rho)\\
&\quad + \gamma(v)\beta(\rho)(\partial_t\rho)^2.
\end{aligned} \tag{3.3}$$

Here we have set

$$\begin{aligned} M_2(\rho) &= M_1(\rho) + M_1^{\mathsf{T}}(\rho) - M_1(\rho)M_1^{\mathsf{T}}(\rho), \\ M_3(\rho) &= (I - M_1(\rho)) : \nabla M_1(\rho), \\ M_4(\rho) &= (I - M_1(\rho))^{\mathsf{T}} M_0(\rho). \end{aligned}$$

(ii) *Maximal regularity of the principal linearized problem.*
First we consider the linear problem defined by the left hand side of (3.2).

$$\left\{\begin{aligned} \kappa_0(x)\partial_t v - d_0(x)\Delta v &= f && \text{in} && \Omega\setminus\Sigma \\ \partial_{\nu_\Omega} v &= 0 && \text{on} && \partial\Omega \\ [\![v]\!] &= 0 && \text{on} && \Sigma \\ l_1(t,x)v + \sigma_0\Delta_\Sigma\rho - \gamma_1(t,x)\partial_t\rho &= g && \text{on} && \Sigma \\ l_0(x)\partial_t\rho - [\![d_0(x)\partial_\nu v]\!] &= h && \text{on} && \Sigma \\ v(0) = v_0,\ \rho(0) &= \rho_0. \end{aligned}\right. \tag{3.4}$$

This inhomogeneous problem can be solved with maximal regularity; see Escher, Prüss, and Simonett [15] for the constant coefficient half-space case with $\gamma \equiv 0$, and Denk, Prüss, and Zacher [9] for the general one-phase case.

Theorem 3.3. ($\gamma \equiv 0$). *Let $p > n+2$, $\sigma > 0$, $\gamma \equiv 0$. Suppose $\kappa_0 \in C(\bar\Omega_j)$ and $d_0 \in C^1(\bar\Omega_j)$, $j = 1, 2$, $\kappa_0, d_0 > 0$ on $\bar\Omega$, $l_0 \in W_p^{2-6/p}(\Sigma)$, and let*

$$l_1 \in W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma))$$

such that $l_0 l_1 > 0$ on $J \times \Sigma$, where $J = [0, t_0]$ is a finite time interval. Then there is a unique solution $z := (v, \rho) \in \mathbb{E}(J)$ of (3.4) if and only if the data (f, g, h) and $z_0 := (v_0, \rho_0)$ satisfy

$$(f,g,h) \in \mathbb{F}(J), \quad z_0 \in [W_p^{2-2/p}(\Omega\setminus\Sigma) \cap C(\bar\Omega)] \times W_p^{4-3/p}(\Sigma),$$

and the compatibility conditions

$$l_1(0)v_0 + \sigma_0\Delta_\Sigma\rho_0 = g(0), \quad h(0) + [\![d_0\partial_\nu v_0]\!] \in W_p^{2-6/p}(\Sigma).$$

The solution map $[(f,g,h,z_0) \mapsto z = (v,\rho)]$ is continuous between the corresponding spaces.

Proof. In the one-phase case this result is proved in [9, Example 3.4]. Therefore, we only indicate the necessary modifications for the two-phase case. The localization procedure can be carried out in the same way as in the one-phase case [9], hence we only need to consider the following model problem with constant coefficients

where the interface is flat:

$$\begin{cases} \kappa_0\partial_t v - d\Delta v = f & \text{in} \quad \dot{\mathbb{R}}^n \\ [\![v]\!] = 0 & \text{on} \quad \mathbb{R}^{n-1} \\ l_1 v + \sigma_0\Delta\rho = g & \text{on} \quad \mathbb{R}^{n-1} \\ l_0\partial_t\rho - [\![d\partial_\nu v]\!] = h & \text{on} \quad \mathbb{R}^{n-1} \\ v(0) = v_0,\ \rho(0) = \rho_0. \end{cases}$$

Here $\dot{\mathbb{R}}^n = \mathbb{R}^{n-1} \times \big(\mathbb{R} \setminus \{0\}\big)$, and $\mathbb{R}^{n-1}$ is identified with $\mathbb{R}^{n-1} \times \{0\}$. Reflecting the lower half-plane to the upper, this becomes a problem of the form studied in [9]. As in Example 3.4 of that paper it is not difficult to verify the necessary Lopatinskii-Shapiro conditions. Then Theorems 2.1 and 2.2 of [9] can be applied, proving the assertion for the model problem. □

Remark 3.4. One might wonder where the somewhat unexpected compatibility condition $h(0) + [\![d_0\partial_\nu v_0]\!] \in W_p^{2-6/p}(\Sigma)$ in the case $\gamma = 0$ comes from. To allude this, note that

$$(h(0) + [\![d_0\partial_\nu v_0]\!])/l_0 = \partial_t\rho(0)$$

is the trace of $\partial_t\rho$ at time $t = 0$. But by the embedding (3.1) this implies that $(h(0) + [\![d_0\partial_\nu v_0]\!])/l_0 \in W_p^{2-6/p}(\Sigma)$, which in turn enforces $h(0) + [\![d_0\partial_\nu u_0]\!] \in W_p^{2-6/p}(\Sigma)$.

The main result for problem (3.4) for $\gamma > 0$ is the following theorem.

Theorem 3.5. ($\gamma > 0$). *Let $p > n + 2$, $\sigma > 0$. Suppose $\kappa_0 \in C(\bar{\Omega}_j)$ and $d_0 \in C^1(\bar{\Omega}_j)$, $j = 1, 2$, $\kappa_0, d_0 > 0$ on $\bar{\Omega}$, $l_0 \in C^1(\Sigma)$, and let*

$$\gamma_1, l_1 \in W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)),$$

such that $\gamma_1 > 0$ on $J \times \Sigma$, where $J = [0, t_0]$ is a finite time interval. Then there is a unique solution $z := (v, \rho) \in \mathbb{E}(J)$ of (3.4) if and only if the data (f, g, h) and $z_0 := (v_0, \rho_0)$ satisfy

$$(f, g, h) \in \mathbb{F}(J), \quad z_0 \in [W_p^{2-2/p}(\Omega \setminus \Sigma) \cap C(\bar{\Omega})] \times W_p^{4-3/p}(\Sigma),$$

and the compatibility conditions

$$(l_0 l_1(0) v_{0|_\Sigma} + l_0\sigma_0\Delta_\Sigma\rho_0 - \gamma_1(0)[\![d\partial_\nu v_0]\!] = \gamma_1(0)h(0) + l_0 g(0).$$

The solution map $[(f, g, h, z_0) \mapsto z = (v, \rho)]$ is continuous between the corresponding spaces.

Proof. The proof of this result is much simpler than for the case $\gamma = 0$. We could follow the strategy in the proof of Theorem 3.3, employing the methods in [9] once more. However, here we want to give a more direct argument that uses the fact that the term $l_0\partial_t\rho$ is of lower order in case $\gamma_1 > 0$. For this purpose, suppose $v_\Sigma := v_{|_\Sigma}$ is known. Consider the problem

$$\gamma_1\partial_t\rho - \sigma_0\Delta_\Sigma\rho = l_1 v_\Sigma - g, \quad t \in J, \quad \rho(0) = \rho_0.$$

Since the Laplace-Beltrami operator is strongly elliptic, we can solve this problem with maximal regularity to obtain ρ in the proper regularity class. Then we solve the transmission problem

$$\left\{\begin{aligned} \kappa_0\partial_t v - d_0\Delta v &= f && \text{in} \quad \Omega\setminus\Sigma \\ \partial_{\nu_\Omega} v &= 0 && \text{on} \quad \partial\Omega \\ [\![v]\!] &= 0 && \text{on} \quad \Sigma \\ -[\![d_0\partial_\nu v]\!] &= h - l_0\partial_t\rho && \text{on} \quad \Sigma \\ v(0) &= v_0. \end{aligned}\right.$$

Finally, we take the trace of v to obtain an equation for v_Σ of the form

$$v_\Sigma = Tv_\Sigma + w,$$

where w is determined by the data alone, and T is a compact operator from $\mathbb{F}_2$ into itself. Here compactness follows from the compact embedding $\mathbb{F}_2 \hookrightarrow \mathbb{F}_3$, i.e. from the regularity of $\partial_t\rho$ which is higher than needed to solve the transmission problem. Thus $I - T$ is a Fredholm operator with index zero, hence invertible since it is injective by causality. This proves the sufficiency of the conditions on the data. Necessity is a consequence of trace theory. □

Remark 3.6. It is interesting to take a look at the boundary symbol of the linear problem. It is of the form

$$s(\lambda,\xi) = \lambda l^2 + (\lambda\gamma + \sigma_0|\xi|^2)[\sqrt{\lambda\kappa_1 + d_1|\xi|^2} + \sqrt{\lambda\kappa_2 + d_2|\xi|^2}].$$

Here $\lambda \in \mathbb{C}_+$ denotes the covariable of time t, and $\xi \in \mathbb{R}^{n-1}$ that of the tangential space variable $x' \in \mathbb{R}^{n-1}$. This symbol is invertible for large λ, provided $\gamma > 0$ or $l \neq 0$. Note that in case $\gamma > 0$ this is a parabolic symbol of order 3/2 in time t and of order 3 in the space variables x. The term λl^2 is of lower order, thus l does not affect well-posedness. On the other hand, for $l = 0$ and $\gamma = 0$ the boundary symbol is ill-posed, since it admits the zeros $(\lambda, 0)$ with arbitrarily large $\operatorname{Re}\lambda$. If $\gamma = 0$ and $l \neq 0$, then it is well-posed. Note that in this case we have order 1 in time, 3 in space, but also the mixed regularity 1/2 in time and 2 in space.

(iii) *Reduction to zero initial values.*
It is convenient to reduce the problem to zero initial data and inhomogeneities with vanishing time trace. This can be achieved as follows. We solve the linear problem (3.4) with initial data v_0, ρ_0 and inhomogenities

$$f = 0, \quad g(t) = e^{\Delta_\Sigma t}G(v_0,\rho_0), \quad h(t) = e^{\Delta_\Sigma t}\rho_1 \text{ with } \rho_1 = H(v_0,\rho_0).$$

Since the Laplace-Beltrami operator Δ_Σ has maximal L_p-regularity, the fact that $G(v_0,\rho_0) \in W_p^{2-3/p}(\Sigma)$ implies $g \in \mathbb{F}_2(J)$. Similarly, $h \in \mathbb{F}_3$ since $\rho_1 \in W_p^{1-3/p}(\Sigma)$. The compatibility conditions yield $[\![d_0\partial_\nu v_0]\!] + \rho_1 \in W_p^{2-6/p}(\Sigma)$. Therefore, the linear problem has a unique solution $z_* := (v_*,\rho_*)$ with maximal regularity $z_* \in$

$\mathbb{E}(J)$. Then we set $\bar{v} = v - v_*$, $\bar{\rho} = \rho - \rho_*$, and obtain the following problem for $\bar{z} = (\bar{v}, \bar{\rho})$.

$$\left\{\begin{aligned} \kappa_0(x)\partial_t\bar{v} - d_0(x)\Delta\bar{v} &= F(\bar{v}+v_*, \bar{\rho}+\rho_*) && \text{in} \quad \Omega\setminus\Sigma \\ \partial_{\nu_\Omega}\bar{v} &= 0 && \text{on} \quad \partial\Omega \\ [\![\bar{v}]\!] &= 0 && \text{on} \quad \Sigma \\ l_1(t,x)\bar{v} + \sigma_0\Delta_\Sigma\bar{\rho} - \gamma_1(t,x)\partial_t\bar{\rho} &= \bar{G}(\bar{v},\bar{\rho}; v_*, \rho_*) && \text{on} \quad \Sigma \\ l_0(x)\partial_t\bar{\rho} - [\![d_0(x)\partial_\nu\bar{v}]\!] &= \bar{H}(\bar{v},\bar{\rho}; v_*, \rho_*) && \text{on} \quad \Sigma \\ \bar{v}(0) = 0,\ \bar{\rho}(0) &= 0. \end{aligned}\right. \tag{3.5}$$

Here we have set

$$\begin{aligned} \bar{G}(\bar{v},\bar{\rho}; v_*, \rho_*) &= G(\bar{v}+v_*, \bar{\rho}+\rho_*) - e^{\Delta_\Sigma t}G(v_0, \rho_0), \\ \bar{H}(\bar{v},\bar{\rho}; v_*, \rho_*) &= H(\bar{v}+v_*, \bar{\rho}+\rho_*) - e^{\Delta_\Sigma t}H(v_0, \rho_0). \end{aligned}$$

Note that $\bar{G}(0,0; v_0, \rho_0) = \bar{H}(0,0; v_0,\rho_0) = 0$ by construction, which ensures time trace zero at $t = 0$.

(iv) *Solution of the nonlinear problem*
We first concentrate on the case $\gamma \equiv 0$, and rewrite problem (3.5) in abstract form as

$$\mathbb{L}\bar{z} = \mathbb{N}(\bar{z}, z_*),$$

where $\mathbb{L} : {}_0\mathbb{E}(0, t_0) \to {}_0\mathbb{F}(0, t_0)$, defined by

$$\mathbb{L}\bar{z} = \big(\kappa_0\partial_t\bar{v} - d_0\Delta\bar{v}, l_1\bar{v} + \sigma_0\Delta_\Sigma\bar{\rho}, l_0\partial_t\bar{\rho} - [\![d_0\partial_\nu\bar{v}]\!]\big),$$

is an isomorphism by Theorem 3.3. The nonlinearity

$$\mathbb{N} : {}_0\mathbb{E}(0, t_0) \times \mathbb{E}(0, t_0) \to {}_0\mathbb{F}(0, t_0),$$

given by the right hand side of(3.5), is of class C^1, since the coefficient functions satisfy $\kappa \in C^1$, $d, l \in C^2$, $\psi \in C^3$, and by virtue of the embeddings

$$\mathbb{E}_1(J) \hookrightarrow C(J \times \bar{\Omega}) \cap C(J; C^1(\bar{\Omega}_j)), \quad \mathbb{E}_2(J) \hookrightarrow C(J; C^3(\Sigma)) \cap C^1(J; C(\Sigma)).$$

Observe that the constants in these embeddings blow up as $t_0 \to 0$, however, they are uniform in t_0 if one considers the space ${}_0\mathbb{E}(J)$!

We want to apply the contraction mapping principle. For this purpose we consider a closed ball $\mathbb{B}_R(0) \subset {}_0\mathbb{E}(0, \tau)$, where the radius $R > 0$ and the final time $\tau \in (0, t_0]$ are at our disposal. We rewrite the abstract equation $\mathbb{L}\bar{z} = \mathbb{N}(\bar{z}, z_*)$ as the fixed point equation

$$\bar{z} = \mathbb{L}^{-1}\mathbb{N}(\bar{z}, z_*) =: \mathbb{T}(\bar{z}), \quad \bar{z} \in \mathbb{B}_R(0).$$

Since we are working in an L_p-setting, by choosing $\tau = \tau(R)$ small enough we can assure that

$$\|\mathbb{T}(0)\|_{\mathbb{E}(0,\tau)} = \|\mathbb{L}^{-1}\mathbb{N}(0, z_*)\|_{\mathbb{E}(0,\tau)} \le R/2.$$

On the other hand, we have

$$\begin{aligned}\|\mathbb{T}(z_1)-\mathbb{T}(z_2)\|_{\mathbb{E}(0,\tau)} \le \|\mathbb{L}^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(0,\tau),{}_0\mathbb{E}(0,\tau))}\times \\ \times \sup_{\|\bar z\|_{{}_0\mathbb{E}(0,\tau)}\le R} \|\mathbb{N}'(\bar z, z_*)\|_{\mathcal{B}({}_0\mathbb{E}(0,\tau),{}_0\mathbb{F}(0,\tau))}\|z_1-z_2\|_{\mathbb{E}(0,\tau)},\end{aligned}$$

hence $\mathbb{T}(\mathbb{B}_R(0))\subset \mathbb{B}_R(0)$ and $\mathbb{T}$ is a strict contraction, provided we have

$$\|\mathbb{L}^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(0,\tau),{}_0\mathbb{E}(0,\tau))} \sup_{\|\bar z\|_{{}_0\mathbb{E}(0,\tau)}\le R} \|\mathbb{N}'(\bar z, z_*)\|_{\mathcal{B}({}_0\mathbb{E}(0,\tau),{}_0\mathbb{F}(0,\tau))} \le 1/2.$$

For this we observe that

$$\|\mathbb{L}^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(0,\tau),{}_0\mathbb{E}(0,\tau))} \le \|\mathbb{L}^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(0,t_0),{}_0\mathbb{E}(0,t_0))} =: C_M < \infty$$

is uniform in $\tau\in(0,t_0)$, since we have vanishing time traces at $t=0$. So it remains to estimate the Frechét-derivative of $\mathbb{N}$ on the ball $\mathbb{B}_R(0)\subset {}_0\mathbb{E}(0,\tau)$. This is the content of the next proposition, which also covers the case $\gamma>0$.

Proposition 3.7. *Let $p>n+2$, $\sigma\in\mathbb{R}$, and suppose $\psi,\gamma\in C^3(0,\infty)$ and $d\in C^2(0,\infty)$. Then $\mathbb{N}: {}_0\mathbb{E}(0,t_0)\times\mathbb{E}(0,t_0)\to {}_0\mathbb{F}(0,t_0)$ is continuously Fréchet-differentiable. There is $\eta>0$ such that for a given $z_*\in\mathbb{E}(0,t_0)$ with $|\rho_0|_{C^2(\Sigma)}\le\eta$, there are continuous functions $\alpha(R)>0$ and $\beta(\tau)>0$ with $\alpha(0)=\beta(0)=0$, such that*

$$\|\mathbb{N}'(\bar z+z_*)\|_{\mathcal{B}({}_0\mathbb{E}(0,\tau),{}_0\mathbb{F}(0,\tau))}\le \alpha(R)+\beta(\tau),\quad \bar z\in\mathbb{B}_R\subset {}_0\mathbb{E}(0,\tau).$$

Proof. We may proceed similarly as in [15, Section 7], where the interface is a graph over $\mathbb{R}^{n-1}$. The additional terms which arise by considering a general geometry are either of lower order or of the form $\tilde M(\bar v,\bar\rho)\nabla_\Sigma\bar\rho$ where $\tilde M(\bar v,\bar\rho)$ is of highest order (see (3.3)), but can be controlled by ensuring that $\nabla_\Sigma\bar\rho$ is sufficiently small. The additional terms due to the presence of γ are of highest order, but small. □

So choosing first $R>0$ and then $\tau>0$ small enough, $\mathbb{T}$ will be a self-map and a strict contraction on $\mathbb{B}_R(0)$. Concluding, the contraction mapping principle yields a unique fixed point $\bar z=\bar z(z_*)\in\mathbb{B}_R(0)\subset {}_0\mathbb{E}(0,\tau)$, hence $z=z_*+\bar z(z_*)$ is the unique solution of (3.2), i.e. of (2.1).

The proof in case $\gamma>0$ is similar, employing now Theorem 3.5.

Remark 3.8. The assumption $p>n+2$ simplifies many arguments since $\mathbb{F}_2(J)$ as well as $\mathbb{F}_3(J)$ are Banach algebras and $\nabla v\in BC(J\times\Omega)$. If we merely assume $p>(n+2)/2$ then $\mathbb{F}_2(J)$ is still a Banach algebra, but $\mathbb{F}_3(J)$ is not, and ∇v may not be bounded anymore. This leads to much more involved estimates for the nonlinearities.

Local semiflows. We denote by $\mathcal{MH}^2(\Omega)$ the closed C^2-hypersurfaces contained in Ω. It can be shown that $\mathcal{MH}^2(\Omega)$ is a C^2-manifold: the charts are the parameterizations over a given hypersurface Σ according to Section 2, and the tangent space consists of the normal vector fields on Σ. We define a metric on $\mathcal{MH}^2(\Omega)$ by means of

$$d_{\mathcal{MH}^2}(\Sigma_1,\Sigma_2):=d_H(\mathcal{N}^2\Sigma_1,\mathcal{N}^2\Sigma_2),$$

where d_H denotes the Hausdorff metric on the compact subsets of $\mathbb{R}^n$ introduced in Section 2. This way $\mathcal{MH}^2(\Omega)$ becomes a Banach manifold of class C^2.

Let $d_\Sigma(x)$ denote the signed distance for Σ as in Section 2. We may then define the *level function* φ_Σ by means of

$$\varphi_\Sigma(x) = \phi(d_\Sigma(x)), \quad x \in \mathbb{R}^n,$$

where

$$\phi(s) = (1 - \chi(s/a)) \operatorname{sgn} s + s\chi(s/a), \quad s \in \mathbb{R}.$$

Then it is easy to see that $\Sigma = \varphi_\Sigma^{-1}(0)$, and $\nabla\varphi_\Sigma(x) = \nu_\Sigma(x)$, for $x \in \Sigma$. Moreover, 0 is an eigenvalue of $\nabla^2\varphi_\Sigma(x)$, and the remaining eigenvalues of $\nabla^2\varphi_\Sigma(x)$ are the principal curvatures of Σ at $x \in \Sigma$.

If we consider the subset $\mathcal{MH}^2(\Omega, r)$ of $\mathcal{MH}^2(\Omega)$ which consists of all closed hypersurfaces $\Gamma \in \mathcal{MH}^2(\Omega)$ such that $\Gamma \subset \Omega$ satisfies a (interior and exterior) ball condition with fixed radius $r > 0$, then the map

$$\Upsilon : \mathcal{MH}^2(\Omega, r) \to C^2(\bar{\Omega}), \quad \Upsilon(\Gamma) := \varphi_\Gamma, \tag{3.6}$$

is an isomorphism of the metric space $\mathcal{MH}^2(\Omega, r)$ onto $\Upsilon(\mathcal{MH}^2(\Omega, r)) \subset C^2(\bar{\Omega})$.

Let $s - (n-1)/p > 2$. Then we define

$$W_p^s(\Omega, r) := \{\Gamma \in \mathcal{MH}^2(\Omega, r) : \varphi_\Gamma \in W_p^s(\Omega)\}. \tag{3.7}$$

In this case the local charts for Γ can be chosen of class W_p^s as well. A subset $A \subset W_p^s(\Omega, r)$ is said to be (relatively) compact, if $\Upsilon(A) \subset W_p^s(\Omega)$ is (relatively) compact.

As an ambient space for the state manifold $\mathcal{SM}_\gamma$ of the Stefan problem with surface tension we consider the product space $C(\bar{G}) \times \mathcal{MH}^2$, due to continuity of temperature and curvature.

We define the state manifolds $\mathcal{SM}_\gamma$, $\gamma \geq 0$, for the Stefan problem (1.1) as follows. For $\gamma = 0$ we set

$$\begin{aligned}\mathcal{SM}_0 := \ &\{(u, \Gamma) \in C(\bar{\Omega}) \times \mathcal{MH}^2 : u \in W_p^{2-2/p}(\Omega \setminus \Gamma),\, \Gamma \in W_p^{4-3/p}, \\ &u > 0 \text{ in } \bar{\Omega},\ [\![\psi(u)]\!] + \sigma\mathcal{H} = 0,\ l(u) \neq 0 \text{ on } \Gamma,\ [\![d\partial_\nu u]\!] \in W_p^{2-6/p}(\Gamma)\},\end{aligned} \tag{3.8}$$

and for $\gamma > 0$

$$\begin{aligned}\mathcal{SM}_\gamma := \ &\{(u, \Gamma) \in C(\bar{\Omega}) \times \mathcal{MH}^2 : u \in W_p^{2-2/p}(\Omega \setminus \Gamma),\, \Gamma \in W_p^{4-3/p}, \\ &u > 0 \text{ in } \bar{\Omega},\ (l(u) - [\![\psi(u)]\!] - \sigma\mathcal{H})([\![\psi(u)]\!] + \sigma\mathcal{H}) = \gamma(u)[\![d\partial_\nu u]\!] \text{ on } \Gamma\}.\end{aligned} \tag{3.9}$$

Charts for these manifolds are obtained by the charts induced by $\mathcal{MH}^2(\Omega)$, followed by a Hanzawa transformation.

Applying Theorem 3.1 or Theorem 3.2, respectively, and re-parameterizing the interface repeatedly, we see that (1.1) yields a local semiflow on $\mathcal{SM}_\gamma$.

Theorem 3.9. *Let $p > n + 2$, $\sigma > 0$ and $\gamma \geq 0$. Then problem (1.1) generates a local semiflow on the state manifold $\mathcal{SM}_\gamma$. Each solution (u, Γ) exists on a maximal time interval $[0, t_*)$, where $t_* = t_*(u_0, \Gamma_0)$.*

Time weights. For later use we need an extension of the local existence results to spaces with time weights. For this purpose, given a UMD-Banach space Y and $\mu \in (1/p, 1]$, we define for $J = (0, t_0)$

$$K^s_{p,\mu}(J;Y) := \{u \in L_{p,loc}(J;Y) : \; t^{1-\mu}u \in K^s_p(J;Y)\},$$

where $s \geq 0$ and $K \in \{H, W\}$. It has been shown in [31] that the operator d/dt in $L_{p,\mu}(J;Y)$ with domain

$$D(d/dt) = {}_0H^1_{p,\mu}(J;Y) = \{u \in H^1_{p,\mu}(J;Y) : \; u(0) = 0\}$$

is sectorial and admits an H^∞-calculus with angle $\pi/2$. However, it does not generate a C_0-semigroup, unless $\mu = 1$. This is the main tool for extending the results for the linear problem, i.e. Theorems 3.3 and 3.5, to the time weighted setting, where the solution space $\mathbb{E}(J)$ is replaced by

$$\mathbb{E}_\mu(J) = \mathbb{E}_{\mu,1}(J) \times \mathbb{E}_{\mu,2}(J),$$

with

$$\begin{aligned}
\mathbb{E}_{\mu,1}(J) &= \{v \in H^1_{p,\mu}(J;L_p(\Omega)) \cap L_{p,\mu}(J;H^2_p(\Omega \setminus \Sigma) : [\![v]\!] = 0, \; \partial_{\nu_\Omega} v = 0\},\\
\mathbb{E}_{\mu,2}(J) &:= W^{3/2-1/2p}_{p,\mu}(J;L_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(J;H^2_p(\Sigma)) \cap L_{p,\mu}(J;W^{4-1/p}_p(\Sigma)), \gamma \equiv 0,\\
\mathbb{E}_{\mu,2}(J) &:= W^{2-1/2p}_{p,\mu}(J;L_p(\Sigma)) \cap L_{p,\mu}(J;W^{4-1/p}_p(\Sigma)), \; \gamma > 0.
\end{aligned}$$

In a similar way, the space of data is defined by

$$\begin{aligned}
\mathbb{F}_{\mu,1}(J) &:= L_{p,\mu}(J;L_p(\Omega)),\\
\mathbb{F}_{\mu,2}(J) &:= W^{1-1/2p}_{p,\mu}(J;L_p(\Sigma)) \cap L_{p,\mu}(J;W^{2-1/p}_p(\Sigma)),\\
\mathbb{F}_{\mu,3}(J) &:= W^{1/2-1/2p}_{p,\mu}(J;L_p(\Sigma)) \cap L_{p,\mu}(J;W^{1-1/p}_p(\Sigma)),\\
\mathbb{F}_\mu(J) &:= \mathbb{F}_{\mu,1}(J) \times \mathbb{F}_{\mu,2}(J) \times \mathbb{F}_{\mu,3}(J).
\end{aligned}$$

The trace spaces for v and ρ for $p > 3$ are then given by

$$v_0 \in W^{2\mu-2/p}_p(\Omega \setminus \Sigma), \quad \rho_0 \in W^{2+2\mu-3/p}_p(\Sigma), \quad \rho_1 \in W^{4\mu-2-6/p}_p(\Sigma), \tag{3.10}$$

where for the last trace - which is of relevance only in case $\gamma \equiv 0$ - we need in addition $\mu > 1/2 + 3/2p$. Note that the embeddings

$$\mathbb{E}_{\mu,1}(J) \hookrightarrow C(J \times \bar{\Omega}) \cap C(J; C^1(\bar{\Omega}_j)), \quad \mathbb{E}_{\mu,2}(J) \hookrightarrow C(J; C^3(\Sigma))$$

require $\mu > 1/2 + (n+2)/2p$, which is feasible since $p > n+2$ by assumption. This restriction is needed for the estimation of the nonlinearities, i.e. Proposition 3.7 remains valid for $\mu \in (1/2 + (n+2)/2p, 1)$.

The assertions for the linear problem remain valid for such μ, replacing $\mathbb{E}(J)$ by $\mathbb{E}_\mu(J)$, $\mathbb{F}(J)$ by $\mathbb{F}_\mu(J)$, for initial data subject to (3.10). This relies on the fact mentioned above that d/dt admits a bounded H^∞-calculus with angle $\pi/2$ in the spaces $L_{p,\mu}(J;Y)$. Therefore the main results in Denk, Prüss and Zacher [9] remain valid for $\mu \in (1/p, 1)$. This has recently been established in [28]. As a consequence of these considerations we have the following result.

Corollary 3.10. *Let $p > n+2$, $\mu \in (1/2 + (n+2)/2p, 1]$, $\sigma > 0$, and suppose that $\psi, \gamma \in C^3(0,\infty)$, $d \in C^2(0,\infty)$ such that $\gamma \equiv 0$ or $\gamma(u) > 0$, $u \in (0,\infty)$, and*

$$\kappa(u) = -u\psi''(u) > 0, \quad d(u) > 0, \quad u \in (0,\infty).$$

Assume the regularity conditions

$$u_0 \in W_p^{2\mu-2/p}(\Omega \setminus \Gamma_0) \cap C(\bar{\Omega}), \quad u_0 > 0, \quad \Gamma_0 \in W_p^{2+2\mu-3/p},$$

and the compatibility conditions

(a) $[\![\psi(u_0)]\!] + \sigma\mathcal{H}(\Gamma_0) = 0$, $[\![d(u_0)\partial_\nu u_0]\!] \in W_p^{4\mu-2-6/p}(\Gamma_0)$, *as well as the* well-posedness condition $l(u_0) \neq 0$ *on* Γ_0, *in case* $\gamma \equiv 0$.

(b) $\big([\![\psi(u_0)]\!] + \sigma\mathcal{H}(\Gamma_0)\big)\big(l(u_0) - [\![\psi(u_0)]\!] - \sigma\mathcal{H}(\Gamma_0)\big) = \gamma(u_0)[\![d(u_0)\partial_\nu u_0]\!]$ *in case* $\gamma > 0$,

Then the transformed problem (2.1) *admits a unique solution $z = (v,\rho) \in \mathbb{E}_\mu(0,\tau)$ for some nontrivial time interval $J = [0,\tau]$. The solution depends continuously on the data. For each $\delta > 0$ the solution belongs to $\mathbb{E}(\delta,\tau)$, i.e. regularizes instantly.*

4. Equilibria

Suppose (u_*, Γ_*) is an equilibrium for (1.1). Then $\partial_t u_* \equiv 0$ as well as $V_* \equiv 0$, and we obtain

$$\left\{\begin{aligned} \operatorname{div}(d(u_*)\nabla u_*) &= 0 && \text{in} && \Omega \setminus \Gamma_* \\ \partial_{\nu_\Omega} u_* &= 0 && \text{on} && \partial\Omega \\ [\![u_*]\!] &= 0 && \text{on} && \Gamma_* \\ [\![\psi(u_*)]\!] + \sigma\mathcal{H}(\Gamma_*) &= 0 && \text{on} && \Gamma_* \\ [\![d(u_*)\partial_\nu u_*]\!] &= 0 && \text{on} && \Gamma_*. \end{aligned}\right. \tag{4.1}$$

This yields $u_* = const$, hence $\mathcal{H}(\Gamma_*) = -[\![\psi(u_*)]\!]/\sigma$ is constant as well. If Γ_* is connected (and Ω is bounded) this implies that Γ_* is a sphere $S_{R_*}(x_0)$ with radius $R_* = \sigma/[\![\psi(u_*)]\!]$. Thus there is an $(n+1)$-parameter family of equilibria

$$\mathcal{E} := \{(u_*, S_{R_*}(x_0)) : u_* > 0,\, 0 < R_* = \sigma/[\![\psi(u_*)]\!],\ \bar{B}_{R_*}(x_0) \subset \Omega\}.$$

Otherwise, Γ_* is the union of finitely many, say m, nonintersecting spheres of equal radius. It will be shown in the proof of Theorem 4.5(vii) that $\mathcal{E}$ is a C^1-manifold of dimension $(mn+1)$ in $W_p^2(\Omega \setminus \Gamma_*) \times W_p^{4-1/p}(\Gamma_*)$.

4.1. **Conservation of Energy.** As we have just seen, the equilibria of (1.1) are constant temperature, and the dispersed phase consists of finitely many non-intersecting balls with the same radius. To determine u and R, taking into account conservation of energy, we have to solve the system

$$\begin{aligned} \mathsf{E}(u,R) := |\Omega_1|\epsilon_1(u) + |\Omega_2|\epsilon_2(u) + \frac{\sigma}{n-1}|\Sigma| &= \mathsf{E}_0, \\ [\![\psi(u)]\!] + \sigma\mathcal{H} &= 0. \end{aligned}$$

In order not to overburden the notation, we use (u,R) instead of (u_*,R_*). The constant E_0 means the initial total energy in the system. Since $\mathcal{H} = -\sigma/R$ we

may eliminate R by the second equation $R = \sigma/[\![\psi(u)]\!]$, and we are left with a single equation for the temperature u:

$$\varphi(u) := \mathsf{E}(u, R(u)) = |\Omega|\epsilon_2(u) - \frac{m\omega_n}{n} R^n(u)[\![\epsilon(u)]\!] + \frac{\sigma m\omega_n}{n-1} R^{n-1}(u) = \varphi_0, \quad (4.2)$$

with $\varphi_0 = \mathsf{E}_0$. Note that only the temperature range $[\![\psi(u)]\!] > 0$ is relevant due to the requirement $R > 0$, and with

$$R_m = \sup\{R > 0 : \Omega \text{ contains } m \text{ disjoint balls of radius } R\}$$

we must also have $R < R_m$, i.e. with $h(u) = [\![\psi(u)]\!]$

$$h(u) > \frac{\sigma}{R_m}.$$

With $\epsilon(u) = \psi(u) - u\psi'(u)$, i.e. $[\![\epsilon(u)]\!] = h(u) - uh'(u)$, we may rewrite $\varphi(u)$ as

$$\varphi(u) = |\Omega|\epsilon_2(u) + c_n\Big(\frac{1}{h(u)^{n-1}} + (n-1)u\frac{h'(u)}{h(u)^n}\Big),$$

where we have set $c_n = m\frac{\omega_n}{n(n-1)}\sigma^n$.

Next, with

$$R'(u) = -\frac{\sigma h'(u)}{h^2(u)} = -\frac{h'(u)R^2(u)}{\sigma}$$

we obtain

$$\begin{aligned}
\varphi'(u) &= |\Omega|\epsilon_2'(u) - [\![\epsilon'(u)]\!]|\Omega_1| + m\omega_n(\frac{\sigma}{R(u)} - [\![\epsilon(u)]\!])R^{n-1}(u)R'(u) \\
&= |\Omega|\kappa_2(u) - [\![\kappa(u)]\!]|\Omega_1| + m\omega_n uh'(u)R^{n-1}(u)R'(u) \\
&= (\kappa(u)|1)_{L_2(\Omega)} - uh'(u)|\Sigma|\frac{h'(u)R^2(u)}{\sigma} \\
&= \Big\{\frac{\sigma u(\kappa(u)|1)_{L_2(\Omega)}}{l^2(u)R^2(u)|\Sigma|} - 1\Big\}\frac{l^2(u)R^2(u)|\Sigma|}{\sigma u},
\end{aligned}$$

with $l(u) = uh'(u)$. It will turn out that in the case of connected phases the term in the parentheses determines whether an equilibrium is stable: it is stable if $\varphi'(u) < 0$ and unstable if $\varphi'(u) > 0$; see Theorem 5.2 below.

In general it is not a simple task to analyze the equation for the temperature

$$\varphi(u) = |\Omega|\epsilon_2(u) + c_n\Big(\frac{1}{h(u)^{n-1}} + (n-1)u\frac{h'(u)}{h(u)^n}\Big) = \varphi_0,$$

unless more properties of the functions $\epsilon_2(u)$ and in particular of $h(u)$ are known. A natural assumption is that h has exactly one positive zero $u_m > 0$, the melting temperature. Therefore we look at two examples.

Example 4.1. Suppose that the heat capacities are identical, i.e. $[\![\kappa]\!] \equiv 0$. This implies

$$uh''(u) = u[\![\psi''(u)]\!] = -[\![\kappa(u)]\!] \equiv 0,$$

which means that $h(u) = h_0 + h_1 u$ is linear. The melting temperature then is $0 < u_m = -h_0/h_1$, hence we have two cases.

Case 1. $h_0 < 0$, $h_1 > 0$; this means $l(u_m) > 0$.
Then the relevant temperature range is $u > u_m$, as h is positive there. We assume now that ϵ_2 is increasing and convex. As $u \to u_m+$ we have $h(u) \to 0$, hence $\varphi(u) \to \infty$, and also $\varphi(u) \to \infty$ for $u \to \infty$ since $\epsilon_2(u)$ is increasing and convex. Further, we have

$$\varphi'(u) = |\Omega|\epsilon_2'(u) - n(n-1)c_n \frac{h_1^2 u}{(h_0 + h_1 u)^{n+1}},$$
$$\varphi''(u) = |\Omega| e_2''(u) + n(n-1)c_n h_1^2 \frac{-h_0 + nh_1 u}{(h_0 + h_1 u)^{n+2}} > 0,$$

which shows that $\varphi(u)$ is strictly convex for $u > u_m$. Thus $\varphi(u)$ has a unique minimum $u_0 > u_m$, $\varphi(u)$ is decreasing for $u_m < u < u_0$ and increasing for $u > u_0$. Thus there are precisely two equilibrium temperatures $u_*^+ \in (u_0, \infty)$ and $u_*^- \in (u_m, u_0)$ provided $\varphi_0 > \varphi(u_0)$ and none if $\varphi_0 < \varphi(u_0)$. The smaller temperature leads to stable equilibria while the larger to unstable ones.

Case 2. $h_0 > 0$, $h_1 < 0$; this means $l(u_m) < 0$.
Then the relevant temperature range is $u < u_m$ as h is positive there. As $u \to u_m-$ we have $h(u) \to 0+$ hence $\varphi(u) \to -\infty$, and as $u \to 0+$ we have $\varphi(u) \to \varphi(0) = |\Omega|\epsilon_2(0) + c_n/h_0^{n-1}$, assuming that $\epsilon_2(0) := \lim_{u\to 0+} \epsilon_2(u)$ exists. Further, for u sufficiently close to zero $\varphi'(u)$ is positive, since $\kappa_2 = \epsilon_2' > 0$, and $\varphi'(u) \to -\infty$ as $u \to u_m-$. Therefore $\varphi'(u)$ admits at least one zero in $(0, u_m)$. But there may be more than one unless $\epsilon_2(u)$ is concave, so let us assume this. Let $u_0 \in (0, u_m)$ denote the absolute maximum of $\varphi(u)$ in $(0, u_m)$. Then there is exactly one equilibrium temperature $u_* \in (u_0, u_m)$ if $\varphi_0 < \varphi(0)$ and it is stable; there are exactly two equilibria $u_*^- \in (0, u_0)$ and $u_*^+ \in (u_0, u_m)$ if $\varphi(0) < \varphi_0 < \varphi(u_0)$, the first one is unstable, the second is stable. If $\varphi_0 > \varphi(u_0)$, there are no equilibria.

Note that in both cases these equilibrium temperatures give rise to equilibria only if the corresponding radius $R(u)$ is smaller than R_m.

Example 4.2. Suppose that the internal energies $\epsilon_i(u)$ are linearly increasing, i.e.

$$\epsilon_i(u) = a_i + \kappa_i u, \quad i = 1, 2,$$

where $\kappa_i > 0$, and now $[\![\kappa]\!] \neq 0$. The identity $\epsilon_i = \psi_i - u\psi_i'$ then leads to

$$\psi_i(u) = a_i + b_i u - \kappa_i u \ln u, \quad i = 1, 2,$$

where the constants b_i are arbitrary. This yields with $\alpha = [\![a]\!]$, $\beta = [\![b]\!]$ and $\delta = [\![\kappa]\!]$

$$h(u) = \alpha + \beta u - \delta u \ln u.$$

Scaling the temperature by $u = u_0 w$ with $\beta - \delta \ln u_0 = 0$ and scaling h we may assume $\beta = 0$ and $\delta = \pm 1$. Then we have to investigate the equation $\varphi(w) = \varphi_1$, where

$$\varphi(w) = cw + \{\frac{1}{h^{n-1}(w)} + (n-1)w\frac{h'(w)}{h^n(w)}\}, \quad h(w) = \pm(\alpha + w \ln w),$$

with $c > 0$ and $\alpha, \varphi_1 \in \mathbb{R}$. The requirement of existence of a melting temperature $w_m > 0$, i.e. a zero of $h(w)$ leads to the restriction $\alpha \le 1/e$.

Actually, the requirement that the melting temperature is unique, i.e. that h has exactly one positive zero implies $\alpha < 0$. Indeed, for $\alpha \in (0, 1/e)$ there is a second zero $w_- > 0$ of h, and h is positive in $(0, w_-)$. Equilibrium temperatures in this range would not make sense physically.

Also here we have to distinguish between two cases, namely that of a plus-sign where the relevant temperature range is $w > w_m$ and in case of a minus-sign it is $(0, w_m)$. Note that h is convex in the first, and concave in the second case.

Case 1: For the derivatives we get in the first case

$$\varphi'(w) = c + (n-1)\left\{\frac{h(w) - nw(h'(w))^2}{h^{n+1}(w)}\right\},$$

$$\varphi''(w) = n(n-1)\frac{h'(w)}{h^{n+2}(w)}\Big\{(n+1)w(h'(w))^2 - h(w)(3 + h'(w))\Big\}.$$

We have $\varphi(w) \to \infty$ for $w \to \infty$ and for $w \to w_m+$, hence $\varphi(w)$ has a global minimum u_0 in (u_m, ∞). Further, $\varphi''(w) > 0$ in (w_m, ∞), hence the minimum is unique and there are precisely two equilibrium temperatures $w_*^- \in (w_m, w_0)$ and $w_*^+ \in (w_0, \infty)$, provided $\varphi_1 > \varphi(w_0)$, the first one is stable, the second unstable.

To prove convexity of φ we write

$$(n+1)w(h'(w))^2 - h(w)(3 + h'(w)) = (n-1)w(h'(w))^2 + f(w),$$

where

$$f(w) = 2w(h'(w))^2 - h(w)(3 + h'(w)) = 2w(1 + \ln w)^2 - (\alpha + w\ln w)(4 + \ln w).$$

We then have $f(w_m) = 2w_m(1 + \ln w_m)^2 > 0$, and

$$f'(w) = (1 + \ln w)^2 + 1 - \alpha/w > 1 - \alpha/w \ge 0,$$

for $\alpha \le 1/e < w_m \le w$. Let us illustrate the sign in h with the water-ice system, ignoring the density jump of water at freezing temperature. So suppose that Ω_2 consists of ice and Ω_1 of water. In this case we have $\kappa_1 > \kappa_2$ hence $\delta < 0$ which implies the plus-sign for h. Here we obtain $w_*^\pm > w_m$, i.e. the ice is overheated. Equilibria only exist if ϕ_1 is large enough, which means that there is enough energy in the system. If the energy in the system is very large then the stable equilibrium temperature w_*^- comes close to the melting temperature w_m and then $R(w)$ will become large, eventually larger than R^*. This excludes equilibria in Ω, the physical interpretation is that everything will eventually melt. On the other hand, if Ω_1 consists of ice and Ω_2 of water, we have the minus sign, which we want to consider next. Here we expect under-cooling of the water-phase, existence of equilibria only for low values of energy, and if the energy in the system is too small everything will freeze.

Case 2: Assume the minus-sign for h and let $\alpha < 0$. Then the relevant temperature range is $(0, w_m)$. Here we have $\varphi(w) \to -\infty$ as $w \to w_m-$ and $\varphi(w) \to 1/|\alpha|^{n-1} > 0$ as $w \to 0+$.

To investigate concavity of φ in the interval $(0, w_m)$, we recompute the derivatives of φ.

$$\varphi'(w) = c - (n-1)\Big\{\frac{1}{h^n(w)} + n\frac{w(h'(w))^2}{h^{n+1}(w)}\Big\},$$
$$\varphi''(w) = n(n-1)\frac{h'(w)}{h^{n+2}(w)}\Big\{(n+1)w(h'(w))^2 + h(w)(3 - h'(w))\Big\}.$$

Setting $w_+ = 1/e$, for $w \in (w_+, w_m)$ we have $h(w) > 0$ and $h'(w) < 0$ hence $\varphi''(w) < 0$. On the other hand, for $w \in (0, w_+)$, both $h(w)$ and $h'(w)$ are positive. Then we rewrite

$$(n+1)w(h'(w))^2 + 3h(w) - h(w)h'(w) = (n-1)w(1 + \ln w)^2 + f(w),$$

where

$$\begin{aligned} f(w) &= 2w(h'(w))^2 + h(w)(3 - h'(w)) \\ &= 2w(1 + \ln w)^2 - (\alpha + w \ln w)(4 + \ln w), \\ f'(w) &= (1 + \ln w)^2 + 1 - \alpha/w > 0, \end{aligned}$$

provided $\alpha \leq 0$. This shows that f is increasing, $f(w) \to -\infty$ as $w \to 0+$, and $f(1/e^3) = 11/e^3 - \alpha > 0$. On the other hand, the function $w(1 + \ln w)^2$ is increasing in $(0, 1/e^3)$, hence $\varphi''(w)$ has a unique zero $w_- \in (0, 1/e^3)$. Therefore φ is concave in $(0, w_-) \cup (w_+, w_m)$ and convex in (w_-, w_+), and φ' has a minimum at w_- and a maximum at w_+. Observe that $\varphi'(w) < c$, $\varphi'(w) \to -\infty$ for $w \to w_m-$ and $\varphi'(0) = c - (n-1)/|\alpha|^n < \varphi'(w_+)$. Therefore, φ' may have no, one, two, or three zeros in $(0, w_m)$ depending on the value of $c > 0$. However, if $c > 0$ is large enough then φ' has only one zero w_1 which lies in (w_+, w_m). In this case φ is increasing in $(0, w_1)$ and decreasing in (w_1, w_m), hence for $\varphi_1 \in (\varphi(0), \varphi(w_1))$ there are precisely two equilibrium temperatures, the smaller leads to unstable, the larger to a stable equilibrium. If $\varphi_1 < \varphi(0)$ there is a unique equilibrium which is stable, and in case $\varphi_1 > \varphi(w_1)$ there is none. However, in general there may be up to four equilibrium temperatures.

4.2. **Linearization at equilibria.** The linearization at an equilibrium (u_*, Γ_*) with $R_* = \sigma/[\![\psi(u_*)]\!]$, reads

$$\left\{\begin{aligned} \kappa_*\partial_t v - d_*\Delta v &= f &&\text{in} && \Omega \setminus \Gamma_* \\ \partial_{\nu_\Omega} v &= 0 &&\text{on} && \partial\Omega \\ [\![v]\!] &= 0 &&\text{on} && \Gamma_* \\ (l_*/u_*)v + \sigma A_*\rho - \gamma_*\partial_t\rho &= g &&\text{on} && \Gamma_* \\ l_*\partial_t\rho - [\![d_*\partial_\nu v]\!] &= h &&\text{in} && \Gamma_* \\ v(0) = v_0,\ \rho(0) &= \rho_0. \end{aligned}\right. \tag{4.3}$$

Here

$$\kappa_* = \kappa(u_*), \quad d_* = d(u_*), \quad l_* = l(u_*), \quad \gamma_* = \gamma(u_*), \quad A_* = \frac{1}{n-1}\big(\frac{n-1}{R_*^2} + \Delta_*\big),$$

where Δ_* denotes the *Laplace-Beltrami* operator on Γ_*.

We note that if $l_* = 0$ and $\gamma_* = 0$ then the problem is not well-posed. On the other hand, if $l_* \neq 0$ and $\gamma_* = 0$, then the operator $-L_0$ defined by

$$\begin{aligned} D(L_0) = \big\{(v,\rho) \in [H_p^2(\Omega\setminus\Gamma_*) \cap C(\bar\Omega)] \times W_p^{4-1/p}(\Gamma_*) :\\ \partial_{\nu_\Omega} v = 0,\ (l_*/u_*)v + \sigma A_*\rho = 0,\ [\![d_*\partial_\nu v]\!] \in W_p^{2-2/p}(\Gamma_*)\big\}, \\ L_0(u,\rho) = ((-d_*/\kappa_*)\Delta v, -[\![(d_*/l_*)\partial_\nu v]\!]), \end{aligned} \tag{4.4}$$

generates an analytic C_0-semigroup with maximal regularity in

$$X_0 := L_p(\Omega) \times W_p^{2-2/p}(\Gamma_*).$$

More precisely, we have the following result.

Theorem 4.3. *Let $3 < p < \infty$, $\sigma > 0$, suppose $\gamma_* = 0$ and let $l_* \neq 0$. Then for each finite interval $J = [0,t_0]$, there is a unique solution $z = (v,\rho) \in \mathbb{E}(J)$ of (4.3) if and only if the data (f,g,h) and $z_0 = (v_0,\rho_0)$ satisfy*

$$(f,g,h) \in \mathbb{F}(J), \quad z_0 \in [W_p^{2-2/p}(\Omega\setminus\Gamma_*) \cap C(\bar\Omega)] \times W_p^{4-3/p}(\Gamma_*)$$

and the compatibility conditions

$$(l_*/u_*)v_0 + \sigma A_*\rho_0 = g(0), \quad h(0) + [\![d_*\partial_\nu v_0]\!] \in W_p^{2-6/p}(\Gamma_*).$$

The operator $-L_0$ defined above generates an analytic C_0-semigroup in X_0 with maximal regularity of type L_p.

In case $\gamma_* > 0$, similar assertions are valid for L_γ in

$$X_\gamma := L_p(\Omega) \times W_p^{2-1/p}(\Gamma_*),$$

where

$$\begin{aligned} D(L_\gamma) = \big\{(v,\rho) \in [H_p^2(\Omega\setminus\Gamma_*) \cap C(\bar\Omega)] \times W_p^{4-1/p}(\Gamma_*) :\\ \partial_{\nu_\Omega} v = 0,\ (l_*^2/u_*)v + l_*\sigma A_*\rho = \gamma_*[\![d_*\partial_\nu v]\!]\big\}, \\ L_\gamma(v,\rho) = ((-d_*/\kappa_*)\Delta v, -(\sigma/\gamma_*)A_*\rho - (l_*/u_*\gamma_*)v). \end{aligned} \tag{4.5}$$

The main result on the problem (4.3) for $\gamma_* > 0$ is the following.

Theorem 4.4. *Let $3 < p < \infty$, and suppose $\sigma, \gamma_* > 0$. Then for each finite interval $J = [0,t_0]$, there is a unique solution $z = (v,\rho) \in \mathbb{E}(J)$ of (4.3) if and only if the data (f,g,h) and $z_0 = (v_0,\rho_0)$ satisfy*

$$(f,g,h) \in \mathbb{F}(J), \quad z_0 \in [W_p^{2-2/p}(\Omega\setminus\Gamma_*) \cap C(\bar\Omega)] \times W_p^{4-3/p}(\Gamma_*)$$

and the compatibility condition

$$(l_*^2/u_*)v_0 + l_*\sigma A_*\rho_0 - \gamma_*[\![d\partial_\nu v_0]\!] = l_*g(0) + \gamma_*h(0).$$

The operator $-L_\gamma$ defined above generates an analytic C_0-semigroup in X_γ with maximal regularity of type L_p.

Proof of Theorem 4.3 and Theorem 4.4. These results are, up to the last assertions, special cases of Theorems 3.3 resp. 3.5 in Section 3. In addition, since the Cauchy problem for L_γ has maximal L_p-regularity, we conclude in both cases by [29, Proposition 1.2] that $-L_\gamma$ generates an analytic C_0-semigroup in X_γ. Recall that the spaces $\mathbb{E}(J)$ are different for $\gamma = 0$ and $\gamma > 0$. □

4.3. **The eigenvalue problem.** By compact embedding, the spectrum of L_γ consists only of countably many discrete eigenvalues of finite multiplicity and is independent of p. The eigenvalue problem reads as follows

$$\left\{\begin{aligned} \kappa_*\lambda v - d_*\Delta v &= 0 && \text{in} && \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v &= 0 && \text{on} && \partial\Omega \\ [\![v]\!] &= 0 && \text{on} && \Gamma_* \\ (l_*/u_*)v + \sigma A_*\rho - \gamma_*\lambda\rho &= 0 && \text{on} && \Gamma_* \\ l_*\lambda\rho - [\![d_*\partial_\nu v]\!] &= 0 && \text{on} && \Gamma_*. \end{aligned}\right. \tag{4.6}$$

Assume first that Γ_* is connected. As shown in [32], $\lambda = 0$ is always an eigenvalue, and $N(L_\gamma)$ is independent of $\gamma_* \geq 0$, κ_* and d_*. We have

$$N(L_\gamma) = \operatorname{span}\left\{(\frac{\sigma u_*}{l_* R_*^2}, -1), (0, Y_1), \ldots, (0, Y_n)\right\}, \tag{4.7}$$

where the functions Y_j denote the *spherical harmonics of degree one*, normalized by $(Y_j|Y_k)_{L_2(\Gamma_*)} = \delta_{jk}$. $N(L_\gamma)$ is isomorphic to the tangent space of $\mathcal{E}$ at $(u_*, \Gamma_*) \in \mathcal{E}$. Let $\lambda \neq 0$ be an eigenvalue with eigenfunction $(v, \rho) \neq 0$. Then (4.6) yields

$$\lambda\big\{|\sqrt{\kappa_*}v|^2_{L_2(\Omega)} - \sigma u_*(A_*\rho|\rho)_{L_2(\Gamma_*)}\big\} + |\sqrt{d_*}\nabla v|^2_{L_2(\Omega)} + \gamma_* u_*|\lambda|^2|\rho|^2_{L_2(\Gamma_*)} = 0.$$

Since A_* is selfadjoint in $L_2(\Gamma_*)$, this identity shows that all eigenvalues of L_γ are real. Decomposing $v = v_0 + \bar{v}$, $\rho = \rho_0 + \bar{\rho}$, with $(\kappa_*|v_0)_{L_2(\Omega)} = (\rho_0|1)_{L_2(\Gamma_*)} = 0$, this identity can be rewritten as

$$\begin{aligned} &\lambda\big\{|\sqrt{\kappa_*}v_0|^2_{L_2(\Omega)} - \sigma u_*(A_*\rho_0|\rho_0)_{L_2(\Gamma_*)} + \lambda u_*\gamma_*|\rho_0|^2_{L_2(\Gamma_*)}\big\} + |\sqrt{d_*}\nabla v_0|^2_{L_2(\Omega)} \\ &+ \big[\lambda\gamma_* u_* + l_*^2|\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)} - \sigma u_*/R_*^2\big]\lambda\bar{\rho}^2|\Gamma_*| = 0. \end{aligned}$$

In the case that Γ_* is connected, the bracket determines whether there is a positive eigenvalue.

If $\Gamma_* = \bigcup_{1\leq l\leq m}\Gamma_*^l$ consists of $m > 1$ spheres Γ_*^l of equal radius, then

$$N(L_\gamma) = \operatorname{span}\left\{(\frac{\sigma u_*}{l_* R_*^2}, -1), (0, Y_1^l), \ldots, (0, Y_n^l) :\, 1 \leq l \leq m\right\}, \tag{4.8}$$

where the functions Y_j^l denote the *spherical harmonics of degree one* on Γ_*^l (and $Y_j^l \equiv 0$ on $\bigcup_{i\neq l}\Gamma_*^i$), normalized by $(Y_j^l|Y_k^l)_{L_2(\Gamma_*^l)} = \delta_{jk}$. $N(L_\gamma)$ is isomorphic to the tangent space of $\mathcal{E}$ at $(u_*, \Gamma_*) \in \mathcal{E}$, as will be shown in Theorem 4.5 below.

Theorem 4.5. *Let $\sigma > 0$, $\gamma_* \geq 0$, $l_* \neq 0$, and assume that the interface Γ_* consists of $m \geq 1$ components. Let*

$$\zeta_* = \frac{\sigma u_* (\kappa_* | 1)_{L_2(\Omega)}}{l_*^2 R_*^2 |\Gamma_*|}, \tag{4.9}$$

and let φ be defined as in (4.2). Then

(i) $\varphi'(u_*) = (\zeta_* - 1) l_*^2 R_*^2 |\Gamma_*| / (\sigma u_*)$.
(ii) 0 *is a an eigenvalue of $-L_\gamma$ with geometric multiplicity $(mn+1)$.*
(iii) 0 *is semi-simple if $\zeta_* \neq 1$.*
(iv) *If Γ_* is connected and $\zeta_* \leq 1$, then all eigenvalues of $-L_\gamma$ are negative, except for 0.*
(v) *If $\zeta_* > 1$, then there are precisely m positive eigenvalues of $-L_\gamma$,*
(vi) *If $\zeta_* \leq 1$ then $-L_\gamma$ has precisely $m-1$ positive eigenvalues.*
(vii) *$N(L_\gamma)$ is isomorphic to the tangent space $T_{(u_*,\Gamma_*)}\mathcal{E}$ of $\mathcal{E}$ at $(u_*, \Gamma_*) \in \mathcal{E}$.*

Remarks 4.6. (a) The result is also true if $l_* = 0$ and $\gamma_* \neq 0$. In this case $\varphi'(u_*) = (\kappa_*|1)_{L_2(\Omega)} > 0$ and $\zeta_* = \infty$, hence the equilibrium is always unstable.

(b) Note that ζ_* does neither depend on d, nor on the undercooling coefficient γ.

(c) For the *Mullins-Sekerka problem*, that is, for $\kappa \equiv 0$, we have $\zeta_* \equiv 0$, in accordance with the result obtained in [13].

(d) It is shown in [32] that in case $\zeta_* = 1$ and Γ_* connected, the eigenvalue 0 is no longer semi-simple: its algebraic multiplicity rises by 1. This is also true if Γ_* is disconnected.

Proof of Theorem 4.5. For the case that Γ_* is connected this result is proved in [32]. The assertions (i)-(iii) also remain valid in the disconnected case. However, the proof of [32, Theorem 2.1(e)], addressing instability, is not completely correct, as it relies on the assertions [32, Proposition 3.2(b) and Proposition 5.1(c)] which are incorrect. (We remark, though, that the instability result of [32, Theorem 1.3] indeed is valid.) Here we give a modified proof for [32, Theorem 2.1(e)] which also applies in case Γ_* is not connected.

It thus remains to prove the assertions in (v), (vi), and (vii). If the stability condition $\zeta_* \leq 1$ does not hold or if Γ_* is disconnected, then there is always a positive eigenvalue. To prove this we proceed as follows. Suppose $\lambda > 0$ is an eigenvalue, and that ρ is known; solve the heat equation

$$\left\{\begin{aligned} \kappa_*\lambda v - d_*\Delta v &= 0 &&\text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v &= 0 &&\text{on} \quad \partial\Omega \\ [\![v]\!] &= 0 &&\text{on} \quad \Gamma_* \\ -[\![d_*\partial_\nu v]\!] &= h &&\text{on} \quad \Gamma_* \end{aligned}\right. \tag{4.10}$$

to get $v = S_\lambda h$, with S_λ being the solution operator. Then taking the trace at Γ_* we obtain $v|_{\Gamma_*} = N_\lambda h$, where N_λ denotes the Neumann-to-Dirichlet operator

for the heat equation (4.10). Setting $h = -\lambda l_* \rho$ this implies with the linearized Gibbs-Thomson law the equation

$$[(l_*^2/u_*)\lambda N_\lambda + \gamma_* \lambda]\rho - \sigma A_* \rho = 0. \tag{4.11}$$

$\lambda > 0$ is an eigenvalue of $-L_\gamma$ if and only if (4.11) admits a nontrivial solution. We consider this problem in $L_2(\Gamma_*)$. Then A_* is selfadjoint and

$$-\sigma(A_* g|g)_{L_2(\Gamma_*)} \geq -\frac{\sigma}{R_*^2}|g|^2_{L_2(\Gamma_*)}.$$

On the other hand, we will see below that N_λ is selfadjoint and positive semi-definite on $L_2(\Gamma_*)$. Moreover, since A_* has compact resolvent, the operator $B_\lambda := [(l_*^2/u_*)\lambda N_\lambda + \gamma_* \lambda] - \sigma A_*$ has compact resolvent as well, for each $\lambda > 0$. Therefore the spectrum of B_λ consists only of eigenvalues which, in addition, are real. We intend to prove that in case either Γ_* is disconnected or the stability condition does not hold, B_{λ_0} has 0 as an eigenvalue, for some $\lambda_0 > 0$.

We will need the following result on the Neumann-to-Dirichlet operator N_λ. We denote by e the function which is identically to one on Γ_*.

Proposition 4.7. *The Neumann-to-Dirichlet operator N_λ for problem* (4.10) *has the following properties in $L_2(\Gamma_*)$.*

(i) *If v denotes the solution of* (4.10)*, then*

$$(N_\lambda h|h)_{L_2(\Gamma_*)} = \lambda|\sqrt{\kappa_*}v|^2_{L_2(\Omega)} + |\sqrt{d_*}\nabla v|^2_{L_2(\Omega)}, \quad \lambda > 0,\ h \in L_2(\Gamma_*).$$

(ii) *For each $\alpha \in (0, 1/2)$ and $\lambda_0 > 0$ there is a constant $C > 0$ such that*

$$(N_\lambda h|h)_{L_2(\Gamma_*)} \geq \frac{\lambda^\alpha}{C}|N_\lambda h|^2_{L_2(\Gamma_*)}, \quad h \in L_2(\Gamma_*),\ \lambda \geq \lambda_0.$$

In particular, N_λ is injective, and

$$|N_\lambda|_{\mathcal{B}(L_2(\Gamma_*))} \leq \frac{C}{\lambda^\alpha}, \quad \lambda \geq \lambda_0.$$

(iii) *On $L_{2,0}(\Gamma_*) = \{\theta \in L_2(\Gamma_*) : (\theta|\mathsf{e})_{L_2(\Gamma_*)} = 0\}$, we even have*

$$(N_\lambda h|h)_{L_2(\Gamma_*)} \geq \frac{(1+\lambda)^\alpha}{C}|N_\lambda h|^2_{L_2(\Gamma_*)}, \quad h \in L_{2,0}(\Gamma_*),\ \lambda > 0,$$

and

$$|N_\lambda|_{\mathcal{B}(L_{2,0}(\Gamma_*), L_2(\Gamma_*))} \leq \frac{C}{(1+\lambda)^\alpha}, \quad \lambda > 0.$$

In particular, for $\lambda = 0$, (4.10) *is solvable if and only if $(h|\mathsf{e})_{\Gamma_*} = 0$, and then the solution is unique up to a constant.*

Proof of Proposition 4.7. The first assertion follows from the divergence theorem. The second and third assertions are consequences of trace theory, combined with Poincaré's inequality. The last assertion is a standard statement in the theory of elliptic transmission problems. We refer to [32]. $\square$

Proof of Theorem 4.5, continued:
(a) Suppose first that Γ_* is connected. Consider $h = \mathsf{e}$. Then with $c_* := l_*^2/u_* \geq 0$ we have

$$(B_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Gamma_*)} = c_*\lambda(N_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Gamma_*)} + \lambda\gamma_*|\mathsf{e}|^2_{L_2(\Gamma_*)} - \frac{\sigma}{R_*^2}|\mathsf{e}|^2_{L_2(\Gamma_*)}.$$

We compute the limit $\lim_{\lambda\to 0}\lambda(N_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Gamma_*)}$ as follows. First solve the problem

$$\left\{\begin{aligned} -d_*\Delta v &= -\kappa_* a_0 && \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v &= 0 && \text{on} \quad \partial\Omega \\ [\![v]\!] &= 0 && \text{on} \quad \Gamma_* \\ -[\![d_*\partial_\nu v]\!] &= \mathsf{e} && \text{on} \quad \Gamma_*, \end{aligned}\right. \tag{4.12}$$

where $a_0 = |\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)}$, which is solvable since the necessary compatibility condition holds. Let v_0 denote the solution which satisfies the normalization condition $(\kappa_*|v_0)_{L_2(\Omega)} = 0$. Then $v_\lambda := S_\lambda \mathsf{e} - v_0 - a_0/\lambda$ satisfies the problem

$$\left\{\begin{aligned} \kappa_*\lambda v_\lambda - d_*\Delta v_\lambda &= -\kappa_*\lambda v_0 && \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v &= 0 && \text{on} \quad \partial\Omega \\ [\![v_\lambda]\!] &= 0 && \text{on} \quad \Gamma_* \\ -[\![d_*\partial_\nu v_\lambda]\!] &= 0 && \text{on} \quad \Gamma_*. \end{aligned}\right. \tag{4.13}$$

By the normalization $(\kappa_*|v_0)_{L_2(\Omega)} = 0$ we see that v_λ is bounded in $W_2^2(\Omega\setminus\Gamma_*)$ as $\lambda\to 0$. Hence we have

$$\lim_{\lambda\to 0}\lambda N_\lambda \mathsf{e} = \lim_{\lambda\to 0}[\lambda v_\lambda|_{\Gamma_*} + \lambda v_0|_{\Gamma_*} + a_0] = a_0 = |\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)}.$$

This then implies

$$\lim_{\lambda\to 0}(B_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Gamma_*)} = c_*\frac{|\Gamma_*|^2}{(\kappa_*|1)_{L_2(\Omega)}} - \frac{\sigma}{R_*^2}|\Gamma_*| < 0,$$

if the stability condition does not hold, i.e. if $\zeta_* > 1$.

(b) Next suppose that Γ_* is disconnected. If Γ_* consists of m components Γ_*^k, $k = 1, ..., m$, we set $\mathsf{e}_k = 1$ on Γ_*^k and zero elsewhere. Let $h = \sum_k a_k \mathsf{e}_k \neq 0$ with $\sum_k a_k = 0$, hence $Q_0 h = h$, where Q_0 is the canonical projection onto $L_{2,0}(\Gamma_*)$,

$$Q_0 h = h - \frac{(h|e)_{L_2(\Gamma_*)}}{|\Gamma_*|}.$$

Then

$$\lim_{\lambda\to 0}\lambda N_\lambda h = \lim_{\lambda\to 0}\lambda N_\lambda Q_0 h = 0,$$

since $N_\lambda Q_0$ is bounded as $\lambda\to 0$. This implies

$$\lim_{\lambda\to 0}(B_\lambda h|h)_{L_2(\Gamma_*)} = -\frac{\sigma}{R_*^2}\sum_k |\Gamma_*^k| a_k^2 < 0.$$

(c) Next we consider the behavior of $(B_\lambda h|h)_{L_2(\Gamma_*)}$ as $\lambda \to \infty$. We want to show that B_λ is positive semi-definite for large λ. For this purpose we introduce the projections P and Q by

$$Ph = c_m \sum_{k=1}^{m} (h|\mathsf{e}_k)_{L_2(\Gamma_*)}\mathsf{e}_k, \quad Q = I - P,$$

where $c_m = m/|\Gamma_*|$ in case Γ_* has m components. Recall that $|\Gamma_*^k| = |\Gamma_*|/m$ for $k = 1, \dots, m$. Then with $h_k = (h|\mathsf{e}_k)_{L_2(\Gamma_*)}$

$$\begin{aligned}|(N_\lambda Ph|Qh)_{L_2(\Gamma_*)}| &\le c_m \sum_k |h_k|\,|(N_\lambda Qh|\mathsf{e}_k)_{L_2(\Gamma_*)}| \\ &\le C\sum_k |h_k||N_\lambda Qh|_{L_2(\Gamma_*)} \le C\lambda^{-\alpha/2}\sum_k |h_k|(N_\lambda Qh|Qh)^{1/2}_{L_2(\Gamma_*)} \\ &\le C\lambda^{-\alpha/2}[\sum_k |h_k|^2 + m(N_\lambda Qh|Qh)_{L_2(\Gamma_*)}] \\ &\le C\lambda^{-\alpha/2}[|Ph|^2_{L_2(\Gamma_*)} + (N_\lambda Qh|Qh)_{L_2(\Gamma_*)}],\end{aligned}$$

for $\lambda > 0$, and C standing for a generic positive constant, which may change from line to line. Hence for $\lambda \ge \lambda_0$, with λ_0 sufficiently large, we have

$$\begin{aligned}(N_\lambda h|h)_{L_2(\Gamma_*)} &= (N_\lambda Qh|Qh)_{L_2(\Gamma_*)} + 2(N_\lambda Qh|Ph)_{L_2(\Gamma_*)} + (N_\lambda Ph|Ph)_{L_2(\Gamma_*)} \\ &\ge \frac{1}{2}(N_\lambda Qh|Qh)_{L_2(\Gamma_*)} + (N_\lambda Ph|Ph)_{L_2(\Gamma_*)} - \frac{C}{\lambda_0^{\alpha/2}}|Ph|^2_{L_2(\Gamma_*)}.\end{aligned}$$

This implies

$$\begin{aligned}(B_\lambda h|h)_{L_2(\Gamma_*)} &= c_*\lambda(N_\lambda h|h)_{L_2(\Gamma_*)} + \gamma_*\lambda|h|^2_{L_2(\Gamma_*)} - (A_*h|h)_{L_2(\Gamma_*)} \\ &\ge \frac{c_*\lambda}{2}(N_\lambda Qh|Qh)_{L_2(\Gamma_*)} + c_*\lambda(N_\lambda Ph|Ph)_{L_2(\Gamma_*)} \\ &\quad - (A_*Qh|Qh)_{L_2(\Gamma_*)} - c|Ph|^2_{L_2(\Gamma_*)}.\end{aligned}$$

Since N_λ is positive semi-definite and also $-A_*Q$ has this property, we only need to prove $\lambda(N_\lambda Ph|Ph)_{L_2(\Gamma_*)} \to \infty$ as $\lambda \to \infty$.

To prove this, similarly as before we estimate

$$|(N_\lambda \mathsf{e}_i|\mathsf{e}_j)_{L_2(\Gamma_*)}| \le C|N_\lambda \mathsf{e}_i|_{L_2(\Gamma_*)} \le \tilde{C}\lambda_0^{-\alpha/2}(N_\lambda \mathsf{e}_i|\mathsf{e}_i)^{1/2}_{L_2(\Gamma_*)},$$

and choosing λ_0 sufficiently large this yields

$$(N_\lambda Pg|Pg)_{L_2(\Gamma_*)} \ge c_0\Big[\min_i (N_\lambda \mathsf{e}_i|\mathsf{e}_i)_{L_2(\Gamma_*)} - \frac{C}{\lambda_0^{\alpha/2}}\Big]|Pg|^2_{L_2(\Gamma_*)}.$$

Therefore it is sufficient to show

$$\lim_{\lambda\to\infty} \lambda(N_\lambda \mathsf{e}_k|\mathsf{e}_k)_{L_2(\Gamma_*)} = \infty, \quad k = 1, \dots, m. \tag{4.14}$$

So suppose, on the contrary, that $\lambda_j(N_\lambda g|g)_{L_2(\Gamma_*)}$ is bounded, for some $g = \mathsf{e}_k$ and some sequence $\lambda_j \to \infty$. Then the corresponding solution v_j of (4.10) is such that

$\lambda_j v_j$ is bounded in $L_2(\Omega)$, hence has a weakly convergent subsequence. W.l.o.g. $\lambda_j v_j \to v_\infty$ weakly in $L_2(\Omega)$. Fix a test function $\psi \in \mathcal{D}(\Omega \setminus \Gamma_*)$. Then

$$\begin{aligned}\lambda_j(\kappa_* v_j|\psi)_{L_2(\Omega)} &= (d_*\Delta v_j|\psi)_{L_2(\Omega)} = (v_j|d_*\Delta\psi)_{L_2(\Omega)}\\ &= (\lambda_j v_j|d_*\Delta\psi)_{L_2(\Omega)}/\lambda_j \to 0\end{aligned}$$

as $j \to \infty$, hence $v_\infty = 0$ in $L_2(\Omega)$. On the other hand we have

$$\begin{aligned}0 < |\Gamma_*|/m &= \int_{\Gamma_*} g d\Gamma_* = \int_{\Gamma_*} -[\![d\partial_\nu v_j]\!] d\Gamma_*\\ &= \int_\Omega d\Delta v_j dx = \lambda_j \int_\Omega \kappa_* v_j dx \to \int_\Omega \kappa_* v_\infty dx,\end{aligned}$$

hence v_∞ is nontrivial, a contradiction. This implies that (4.14) is valid, provided $l_* > 0$.

On the other hand, in case $l_* = 0$ we have $\gamma_* > 0$, hence $\lambda\gamma_*|g|^2_{L_2(\Gamma_*)} \to \infty$, so also in this case B_λ is positive semi-definite for large λ.

(d) Summarizing, we have shown that B_λ is not positive semi-definite for small $\lambda > 0$ if either Γ_* is not connected or the stability condition does not hold, and B_λ is always positive semi-definite for large λ. Set

$$\lambda_0 = \sup\{\lambda > 0 : B_\mu \text{ is not positive semi-definite for each } \mu \in (0, \lambda]\}.$$

Since B_λ has compact resolvent, B_λ has a negative eigenvalue for each $\lambda < \lambda_0$. This implies that 0 is an eigenvalue of B_{λ_0}, thereby proving that $-L_\gamma$ admits the positive eigenvalue λ_0.

Moreover, we have also shown that

$$B_0 h = \lim_{\lambda\to 0} c_*\lambda N_\lambda h - \sigma\mathcal{A}_* h = c_*|\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)} P_0 h - \sigma\mathcal{A}_* h,$$

where $P_0 h := (I - Q_0)h = (h|\mathsf{e})_{L_2(\Gamma_*)}/|\Gamma_*|$. Therefore, B_0 has the eigenvalue $c_*|\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)} - \sigma/R_*^2$ with eigenfunction e, and in case $m > 1$ it also possesses the eigenvalue $-\sigma/R_*^2$ with precisely $m-1$ linearly independent eigenfunctions of the form $\sum_k a_k \mathsf{e}_k$ with $\sum_k a_k = 0$. This implies that $-L_\gamma$ has exactly m positive eigenvalues if the stability condition does not hold, and $m - 1$ otherwise.

(e) It remains to show assertion (vii). Suppose for the moment that Γ_* consists of a single sphere of radius $R_* = \sigma/[\![\psi(u_*)]\!]$, centered at the origin of $\mathbb{R}^n$. Suppose $\mathcal{S} \subset \Omega$ is a sphere that is sufficiently close to Γ_*. Denote by $(z_1, \ldots, z_n)$ the coordinates of its center and let z_0 be such that $\sigma/[\![\psi(u_* + z_0)]\!]$ corresponds to its radius. Then, by [13, Section 6], the sphere $\mathcal{S}$ can be parameterized over Γ_* by the distance function

$$\rho(z) = \sum_{j=1}^n z_j Y_j - R_* + \sqrt{(\sum_{j=1}^n z_j Y_j)^2 + (\sigma/[\![\psi(u_* + z_0)]\!])^2 - \sum_{j=1}^n z_j^2}.$$

Denoting by O a sufficiently small neighborhood of 0 in $\mathbb{R}^{n+1}$, the mapping

$$[z \mapsto \Psi(z) := (u_* + z_0, \rho(z))] : O \to W^2_p(\Omega) \times W^{4-1/p}_p(\Gamma_*)$$

is C^1 (in fact C^k if ψ is C^k), and the derivative at 0 is given by

$$\Psi'(0)h = \big(1, -\sigma[\![\psi'(u_*)]\!]/[\![\psi(u_*)]\!]^2\big)h_0 + \big(0, \sum_{j=1}^{n} h_j Y_j\big), \quad h \in \mathbb{R}^{n+1}.$$

Noting that $\sigma[\![\psi'(u_*)]\!]/[\![\psi(u_*)]\!]^2 = l_* R_*^2/(\sigma u_*)$ we can conclude that near (u_*, Γ_*) the set $\mathcal{E}$ of equilibria is a C^1-manifold in $W_p^2(\Omega) \times W_p^{4-1/p}(\Gamma_*)$ of dimension $n+1$, and that the tangent space $T_{(u_*,\Gamma_*)}(\mathcal{E})$ coincides with $N(L_\gamma)$, see (4.7). It is now easy to see that this result remains valid for the case of m spheres of the same radius R_*. The dimension of $\mathcal{E}$ is then given by $(mn+1)$, as mn parameters are needed to locate the respective centers, and one additional parameter is needed to track the temperature. □

5. Nonlinear Stability and Instability of Equilibria

Before we discuss nonlinear stability of equilibria, we need some preparations. The first observation is that the equations near an equilibrium $(u_*, \Gamma_*) \in \mathcal{E}$ can be restated as

$$\left\{\begin{aligned} \kappa_*\partial_t v - d_*\Delta v &= F_*(v,\rho) && \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v &= 0 && \text{on} \quad \partial\Omega \\ [\![v]\!] &= 0 && \text{on} \quad \Gamma_* \\ (l_*/u_*)v + \sigma A_*\rho - \gamma_*\partial_t\rho &= G_*(v,\rho) && \text{on} \quad \Gamma_* \\ l_*\partial_t\rho - [\![d_*\partial_\nu v]\!] &= H_*(v,\rho) && \text{on} \quad \Gamma_* \\ v(0) = v_0,\ \rho(0) &= \rho_0. \end{aligned}\right. \tag{5.1}$$

where

$$\begin{aligned} F_*(v,\rho) &= (\kappa_* - \kappa(u_*+v))\partial_t v + (d(u_*+v) - d_*)\Delta v + d(u_*+v)M_2(\rho) : \nabla^2 v \\ &\quad - d'(u_*+v)|(I - M_1(\rho))\nabla v|^2 + d(u_*+v)(M_3(\rho)|\nabla v) \\ &\quad + \kappa(u_*+v)\mathcal{R}(\rho)(u_*+v), \\ G_*(v,\rho) &= -([\![\psi(u_*+v)]\!] + \sigma\mathcal{H}(\rho)) + (l_*/u_*)v + \sigma A_*\rho + (\gamma(u_*+v)\beta(\rho) - \gamma_*)\partial_t\rho, \\ H_*(v,\rho) &= [\![(d(u_*+v) - d_*)\partial_\nu v]\!] + (l_* - l(u_*+v))\partial_t\rho \\ &\quad + ([\![d(u_*+v)\nabla v]\!]|M_4(\rho)\nabla_\Sigma\rho) + \gamma(u_*+v)\beta(\rho)(\partial_t\rho)^2, \end{aligned}$$

see Sections 2 and 3 for the definition of $M_j(\rho)$, $j = 1, \dots, 4$. The nonlinearities satisfy $F_*(0,0) = G_*(0,0) = H_*(0,0) = 0$ as well as $F_*'(0,0) = G_*'(0,0) = H_*'(0,0) = 0$.

The state manifold for problem (5.1) near the equilibrium (u_*, Γ_*) can then be described by

$$\begin{aligned}\mathcal{SM}_0 = \big\{(v,\rho) \in W_p^{2-2/p}(\Omega\setminus\Gamma_*) \times W_p^{4-3/p}(\Gamma_*):\ \partial_{\nu_\Omega} v = 0,\ [\![v]\!] = 0,\\ (l_*/u_*)v + \sigma A_*\rho = G_*(v,\rho),\ [\![d_*\partial_\nu v]\!] + H_*(v,\rho) \in W_p^{2-6/p}(\Gamma_*)\big\},\end{aligned} \tag{5.2}$$

for $\gamma_* = 0$, in case $l_* \neq 0$ (otherwise the linear problem is not well-posed) and

$$\begin{aligned}\mathcal{SM}_\gamma = \big\{(v,\rho) \in W_p^{2-2/p}(\Omega\setminus\Gamma_*) \times W_p^{4-3/p}(\Gamma_*):\ \partial_{\nu_\Omega} v = 0,\ [\![v]\!] = 0,\\ (l_*^2/u_*)v + l_*\sigma A_*\rho - \gamma_*[\![d_*\partial_\nu v]\!] = l_* G_*(v,\rho) + \gamma_* H_*(v,\rho)\big\},\end{aligned} \tag{5.3}$$

in case $\gamma_* > 0$.

We would like to parameterize these manifolds over their tangent spaces at $(0,0)$, given by

$$\begin{aligned}\tilde Z_0 = \big\{(\tilde v,\tilde\rho) \in [W_p^{2-2/p}(\Omega\setminus\Gamma_*) \cap C(\bar\Omega)] \times W_p^{4-3/p}(\Gamma_*):\\ \partial_{\nu_\Omega}\tilde v = 0,\ (l_*/u_*)\tilde v + \sigma A_*\tilde\rho = 0,\ [\![d_*\partial_\nu \tilde v]\!] \in W_p^{2-6/p}(\Gamma_*)\big\},\end{aligned} \tag{5.4}$$

respectively, for $\gamma_* > 0$

$$\begin{aligned}\tilde Z_\gamma = \big\{(\tilde v,\tilde\rho) \in [W_p^{2-2/p}(\Omega\setminus\Gamma_*) \cap C(\bar\Omega)] \times W_p^{4-3/p}(\Gamma_*):\\ \partial_{\nu_\Omega}\tilde v = 0,\ (l_*^2/u_*)\tilde v + l_*\sigma A_*\tilde\rho - \gamma_*[\![d_*\partial_\nu \tilde v]\!] = 0\big\}.\end{aligned} \tag{5.5}$$

Note that the norm in $\tilde Z_\gamma$ for $\gamma = 0$ is given by

$$|(\tilde v,\tilde\rho)|_{\tilde Z_0} = |\tilde v|_{W_p^{2-2/p}} + |\tilde\rho|_{W_p^{4-3/p}} + |[\![d_*\partial_\nu\tilde v]\!]|_{W_p^{2-6/p}},$$

while for $\gamma > 0$ it is given by $|(\tilde v,\tilde\rho)|_{\tilde Z_\gamma} = |\tilde v|_{W_p^{2-2/p}} + |\tilde\rho|_{W_p^{4-3/p}}$.

In order to determine a parameterization, we consider the linear problem

$$\left\{\begin{aligned} \kappa_*\omega v - d_*\Delta v &= 0 && \text{in} && \Omega\setminus\Gamma_*\\ \partial_{\nu_\Omega} v &= 0 && \text{on} && \partial\Omega\\ [\![v]\!] &= 0 && \text{on} && \Gamma_*\\ (l_*/u_*)v + \sigma A_*\rho - \gamma_*\omega\rho &= g && \text{on} && \Gamma_*\\ l_*\omega\rho - [\![d_*\partial_\nu v]\!] &= h && \text{on} && \Gamma_*.\end{aligned}\right. \tag{5.6}$$

We have the following result.

Proposition 5.1. *Suppose $p > 3$, $\gamma_* \geq 0$, $l_* \neq 0$ in case $\gamma_* = 0$, and $\omega > 0$ is sufficiently large. Then problem (5.6) admits a unique solution (v,ρ) with regularity*

$$v \in W_p^{2-2/p}(\Omega\setminus\Gamma_*), \quad \rho \in W_p^{4-3/p}(\Gamma_*)$$

if and only if the data (g,h) satisfy

$$g \in W_p^{2-3/p}(\Gamma_*), \quad h \in W_p^{1-3/p}(\Gamma_*).$$

The solution map $[(g,h) \mapsto (v,\rho)]$ is continuous in the corresponding spaces.

Proof. This purely elliptic problem can be solved in the same way as the corresponding linear parabolic problems, cf. Theorems 4.3 and 4.4. □

For the parametrization we pick $\omega > 0$ sufficiently large. Given $\tilde{z} = (\tilde{v}, \tilde{\rho}) \in \tilde{Z}_\gamma$ sufficiently small, we can solve the auxiliary problem

$$\left\{\begin{aligned} \kappa_*\omega\bar{v} - d_*\Delta\bar{v} &= 0 && \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega}\bar{v} &= 0 && \text{on} \quad \partial\Omega \\ [\![\bar{v}]\!] &= 0 && \text{on} \quad \Gamma_* \\ (l_*/u_*)\bar{v} + \sigma A_*\bar{\rho} - \gamma_*\omega\bar{\rho} &= G_*(\tilde{v}+\bar{v}, \tilde{\rho}+\bar{\rho}) && \text{on} \quad \Gamma_* \\ l_*\omega\bar{\rho} - [\![d_*\partial_\nu\bar{v}]\!] &= H_*(\tilde{v}+\bar{v}, \tilde{\rho}+\bar{\rho}) && \text{on} \quad \Gamma_* \end{aligned}\right. \tag{5.7}$$

by means of the implicit function theorem, employing Proposition 5.1. This yields a unique solution $\bar{z} = (\bar{v}, \bar{\rho}) = \phi(\tilde{z}) \in W_p^{2-2/p}(\Omega\setminus\Gamma_*) \times W_p^{4-3/p}(\Gamma_*)$ with a C^1-function ϕ such that $\phi(0) = 0$ as well as $\phi'(0) = 0$. One readily verifies that $z = \tilde{z} + \phi(\tilde{z}) \in \mathcal{SM}_\gamma$. To prove surjectivity of this map, given $(v, \rho) \in \mathcal{SM}_\gamma$, we solve the linear problem

$$\left\{\begin{aligned} \kappa_*\omega\bar{v} - d_*\Delta\bar{v} &= 0 && \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega}\bar{v} &= 0 && \text{on} \quad \partial\Omega \\ [\![\bar{v}]\!] &= 0 && \text{on} \quad \Gamma_* \\ (l_*/u_*)\bar{v} + \sigma A_*\bar{\rho} - \gamma_*\omega\bar{\rho} &= G_*(v, \rho) && \text{on} \quad \Gamma_* \\ l_*\omega\bar{\rho} - [\![d_*\partial_\nu\bar{v}]\!] &= H_*(v, \rho) && \text{on} \quad \Gamma_* \end{aligned}\right. \tag{5.8}$$

and set $\tilde{z} = z - \bar{z}$. Then $\tilde{z} \in \tilde{Z}_\gamma$ and $\bar{z} = \phi(\tilde{z})$, hence the map $[\tilde{z} \mapsto \tilde{z} + \phi(\tilde{z})]$ is also surjective near 0. We have, thus, obtained a local parametrization of $\mathcal{SM}_\gamma$ near zero over the tangent space $\tilde{Z}_\gamma$.

Next we derive a similar decomposition for the solutions of problem (5.1). Let $z_0 = (\tilde{z}_0, \phi(\tilde{z}_0)) \in \mathcal{SM}_\gamma$ be given and let $z \in \mathbb{E}(a)$ be the solution of (5.1) with initial value z_0. Then we would like to devise a decomposition $z = z_\infty + \tilde{z} + \bar{z}$, where $\tilde{z}(t) \in \tilde{Z}_\gamma$ for all $t \in [0, a]$, and where $z_\infty = \tilde{z}_\infty + \phi(\tilde{z}_\infty)$ is an equilibrium for (5.1). In order to achieve this, we consider the coupled systems of equations

$$\left\{\begin{aligned} \kappa_*\omega\bar{v} + \kappa_*\partial_t\bar{v} - d_*\Delta\bar{v} &= F_*(z_\infty + \tilde{z} + \bar{z}) - F_*(z_\infty) \\ \partial_{\nu_\Omega}\bar{v} &= 0 \\ [\![\bar{v}]\!] &= 0 \\ (l_*/u_*)\bar{v} + \sigma A_*\bar{\rho} - \gamma_*(\partial_t\bar{\rho} + \omega\bar{\rho}) &= G_*(z_\infty + \tilde{z} + \bar{z}) - G_*(z_\infty) \\ l_*\omega\bar{\rho} + l_*\partial_t\bar{\rho} - [\![d_*\partial_\nu\bar{v}]\!] &= H_*(z_\infty + \tilde{z} + \bar{z}) - H_*(z_\infty) \\ \bar{z}(0) &= \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty), \end{aligned}\right. \tag{5.9}$$

and

$$\left\{\begin{aligned} \kappa_*\partial_t\tilde v - d_*\Delta\tilde v &= \kappa_*\omega\bar v\\ \partial_{\nu_\Omega}\tilde v &= 0\\ [\![\tilde v]\!] &= 0\\ (l_*/u_*)\tilde v + \sigma A_*\tilde\rho - \gamma_*\partial_t\tilde\rho &= -\gamma_*\omega\bar\rho\\ l_*\partial_t\tilde\rho - [\![d_*\partial_\nu\tilde v]\!] &= l_*\omega\bar\rho\\ \tilde z(0) &= \tilde z_0 - \tilde z_\infty. \end{aligned}\right. \tag{5.10}$$

It should be mentioned that $F_*(z_\infty) = 0$, as can be seen from the equilibrium equation for (5.1) and the fact that v_∞ = constant for $z_\infty = (v_\infty, \rho_\infty)$. For reasons of symmetry and consistency we will, nevertheless, include this term.

Equations (5.9)–(5.10) can be rewritten in the more condensed form

$$\begin{aligned} &\mathbb{L}_{\gamma,\omega}\bar z = N(z_\infty + \tilde z + \bar z) - N(z_\infty), && \bar z(0) = \phi(\tilde z_0) - \phi(\tilde z_\infty),\\ &\dot{\tilde z} + L_\gamma\tilde z = \omega\bar z, && \tilde z(0) = \tilde z_0 - \tilde z_\infty, \end{aligned} \tag{5.11}$$

where we use the abbreviation $\mathbb{L}_{\gamma,\omega}$ to denote the linear operator on the left hand side of (5.9), and N to denote the nonlinearities on the right hand side of (5.9), respectively. We are now ready to formulate the main theorem of this section.

Theorem 5.2. *Suppose $\sigma > 0$, $\gamma_* = \gamma(u_*) \geq 0$ and $l_* = l(u_*) \neq 0$ in case $\gamma_* = 0$. Then in the topology of the state manifold $\mathcal{SM}_\gamma$ we have:*

(a) *$(u_*, \Gamma_*) \in \mathcal{E}$ is stable if Γ_* is connected and $\zeta_* < 1$.*
Any solution starting in a neighborhood of such a stable equilibrium exists globally and converges to another stable equilibrium exponentially fast.

(b) *$(u_*, \Gamma_*) \in \mathcal{E}$ is unstable if Γ_* is disconnected or if $\zeta_* > 1$.*
Any solution starting and staying in a neighborhood of such an unstable equilibrium converges to another unstable equilibrium exponentially fast.

Proof. **(a)** We begin with the case that (u_*, Γ_*) is linearly stable. Then according to Theorem 4.5 we have $X_\gamma = N(L_\gamma) \oplus R(L_\gamma)$. Let P^c denote the projection onto $X^c_\gamma := N(L_\gamma)$ along $X^s_\gamma := R(L_\gamma)$ and $P^s = I - P^c$ the complementary projection onto $R(L_\gamma)$. We parameterize the set of equilibria $\mathcal{E}$ near 0 over $N(L_\gamma)$ via the C^1-map $[\mathsf{x} \mapsto \mathsf{x} + \psi(\mathsf{x}) + \phi(\mathsf{x} + \psi(\mathsf{x}))]$ such that $\psi(0) = \psi'(0) = 0$ and $\phi(0) = \phi'(0) = 0$. It follows from the equilibrium equation associated to (5.1) (recall that $F_*(z_e)$ vanishes for any equilibrium z_e), and from the definition of ϕ that the mapping ψ is determined by the equation

$$L^s_\gamma\psi(\mathsf{x}) = P^s\omega\phi(\mathsf{x} + \psi(\mathsf{x})), \quad \mathsf{x} \in B_{X^c_\gamma}(r). \tag{5.12}$$

Since L^s_γ is invertible on X^s_γ, $\psi \in C^1(B_{X^c_\gamma}(r), D(L^s_\gamma))$ is well-defined by the implicit function theorem and $\psi(0) = \psi'(0) = 0$.

For $\mathsf{x}_\infty \in X^c_\gamma$ sufficiently small we set $z_\infty := \mathsf{x}_\infty + \psi(\mathsf{x}_\infty) + \phi(\mathsf{x}_\infty + \psi(\mathsf{x}_\infty))$. Then z_∞ is an equilibrium for (5.1) and we will now consider the decomposition

$z = z_\infty + \tilde{z} + \bar{z}$ introduced in (5.9)–(5.10), or (5.11), respectively. With the ansatz

$$\tilde{z} = \mathsf{x} + \psi(\mathsf{x}_\infty + \mathsf{x}) - \psi(\mathsf{x}_\infty) + \mathsf{y}, \quad (\mathsf{x}, \mathsf{y}) \in X^c_\gamma \times X^s_\gamma, \tag{5.13}$$

for $\mathsf{x}, \mathsf{x}_\infty \in X^c_\gamma$ small enough, the second line in (5.11) becomes

$$\begin{cases} \dot{\mathsf{x}} = P^c \omega \bar{z}, & \mathsf{x}(0) = \mathsf{x}_0 - \mathsf{x}_\infty, \\ \dot{\mathsf{y}} + L^s_\gamma \mathsf{y} = S(\mathsf{x}_\infty, \mathsf{x}, \bar{z}), & \mathsf{y}(0) = \mathsf{y}_0, \end{cases} \tag{5.14}$$

where

$$S(\mathsf{x}_\infty, \mathsf{x}, \bar{z}) = P^s \omega \bar{z} - \psi'(\mathsf{x}_\infty + \mathsf{x}) P^c \omega \bar{z} - L^s_\gamma [\psi(\mathsf{x}_\infty + \mathsf{x}) - \psi(\mathsf{x}_\infty)],$$

and

$$\tilde{z}_0 = \mathsf{x}_0 + \psi(\mathsf{x}_0) + \mathsf{y}_0, \quad (\mathsf{x}_0, \mathsf{y}_0) \in X^c_\gamma \times (X^s_\gamma \cap \tilde{Z}_\gamma). \tag{5.15}$$

Next we show that the system of equations (5.14) admits a unique global solution $(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y})$, where the functions (x, y) are exponentially decaying, provided $\bar{z}$ is exponentially decaying and $(\mathsf{x}_0, \mathsf{y}_0)$ is sufficiently small. For this let us first introduce some more notation. For $\delta \geq 0$ we set

$$\begin{aligned} \mathbb{E}_i(\mathbb{R}_+, \delta) &:= \{v : e^{\delta t} v \in \mathbb{E}_i(\mathbb{R}_+)\}, \quad i = 1, 2, \\ \mathbb{F}_j(\mathbb{R}_+, \delta) &:= \{v : e^{\delta t} v \in \mathbb{F}_j(\mathbb{R}_+)\}, \quad j = 1, 2, 3, \end{aligned}$$

endowed with the norms

$$\|v\|_{\mathbb{E}_i(\mathbb{R}_+, \delta)} = \|e^{\delta t} v\|_{\mathbb{E}_i(\mathbb{R}_+)}, \quad \|v\|_{\mathbb{F}_j(\mathbb{R}_+, \delta)} = \|e^{\delta t} v\|_{\mathbb{F}_j(\mathbb{R}_+)}.$$

The spaces $\mathbb{E}(\mathbb{R}_+, \delta)$ and $\mathbb{F}(\mathbb{R}_+, \delta)$ are then defined analogously as in Section 3. We also need the space

$$\mathbb{X}(\mathbb{R}_+, \delta) := H^1_p(\mathbb{R}_+, \delta; X_\gamma) \cap L_p(\mathbb{R}_+, \delta; D(L_\gamma)), \tag{5.16}$$

where $H^k_p(\mathbb{R}_+, \delta; E)$ denotes all functions $v : \mathbb{R}_+ \to E$ such that $e^{\delta t} v \in H^k_p(\mathbb{R}_+; E)$, with E a given Banach space. Finally, let

$$\mathbb{B}_1(r, \delta) : = \{(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in X^c_\gamma \times (X^s_\gamma \cap \tilde{Z}_\gamma) \times \mathbb{E}(\mathbb{R}_+, \delta) : |(\mathsf{x}_0, \mathsf{y}_0)|_{\tilde{Z}_\gamma} < r\}.$$

For given $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0, \delta)$, with r_0 sufficiently small, we set

$$\begin{aligned} \mathsf{x}_\infty &:= \mathsf{x}_0 + \int_0^\infty P^c \omega \bar{z}(\tau)\, d\tau, \\ \mathsf{x} &:= -\int_t^\infty P^c \omega \bar{z}(\tau)\, d\tau, \\ \mathsf{y} &:= \left(\frac{d}{dt} + L^s_\gamma, \mathrm{tr}\right)^{-1} \big(S(\mathsf{x}_\infty, \mathsf{x}, \bar{z}), \mathsf{y}_0\big). \end{aligned} \tag{5.17}$$

Here we used the notation $\mathrm{tr}\, w := w(0)$. It should be observed that the functions $(\mathsf{x}_\infty, \mathsf{x})$ occurring in the third line of (5.17) are defined through the first two lines in (5.17). We now set

$$\mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := (\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}), \quad (\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0, \delta), \tag{5.18}$$

where r_0 is chosen sufficiently small.

Next we will show that there exists a number $\delta_0 > 0$ such that for any $\delta \in [0, \delta_0]$ the mapping $\mathfrak{S}$ has the following properties:

$$\mathfrak{S} \in C\big(\mathbb{B}_1(r_0,\delta), X_\gamma^c \times \mathbb{X}^c(\mathbb{R}_+,\delta) \times \mathbb{X}^s(\mathbb{R}_+,\delta)\big), \quad \mathfrak{S}(0) = 0, \tag{5.19}$$

where

$$\mathbb{X}^c(\mathbb{R}_+,\delta) := H_p^1(\mathbb{R}_+,\delta; X_\gamma^c), \quad \mathbb{X}^s(\mathbb{R}_+,\delta) := H_p^1(\mathbb{R}_+,\delta; X_\gamma^s) \cap L_p(\mathbb{R}_+,\delta; D(L_\gamma^s)).$$

Writing $\mathfrak{S} = (\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3)$ we readily observe that

$$\mathfrak{S}_1 \in C^\infty(\mathbb{B}_1(r_0,\delta), X_\gamma^c), \quad \mathfrak{S}_1(0) = 0. \tag{5.20}$$

For $g \in L_p(\mathbb{R}_+,\delta; X_\gamma^c)$, let $(Kg)(t) := \int_t^\infty g(\tau)\,d\tau$ and note that

$$e^{\delta t}(Kg)(t) = \int_t^\infty e^{\delta(t-\tau)} e^{\delta\tau} g(\tau)\,d\tau.$$

Young's inequality for convolution integrals readily yields

$$K \in \mathcal{B}\big(L_p(\mathbb{R}_+,\delta; X_\gamma^c), H_p^1(\mathbb{R}_+,\delta; X_\gamma^c)\big),$$

and this shows that $\mathfrak{S}_2 \in \mathbb{X}^c(\mathbb{R}_+,\delta)$. Hence we have

$$\mathfrak{S}_2 \in C^\infty\big(\mathbb{B}_1(r_0,\delta), \mathbb{X}^c(\mathbb{R}_+,\delta)\big), \quad \mathfrak{S}_2(0) = 0. \tag{5.21}$$

Concerning the function $\mathfrak{S}_3$, we know from Theorem 4.5(v) that $s(-L_\gamma^s)$, the spectral bound of $(-L_\gamma^s)$, is negative. Fixing $\delta_0 > 0$ with $s(-L_\gamma^s) < -\delta_0$ it follows from semigroup theory and the L_p-maximal regularity results stated in Theorem 4.3 and Theorem 4.4 that

$$(\frac{d}{dt} + L_\gamma^s, \mathrm{tr})^{-1} \in \mathcal{B}\big(L_p(\mathbb{R}_+,\delta; X_\gamma^s) \times \tilde{Z}_\gamma^s,\, \mathbb{X}^s(\mathbb{R}_+,\delta)\big), \quad \delta \in [0,\delta_0], \tag{5.22}$$

where $\tilde{Z}_\gamma^s = X_\gamma^s \cap \tilde{Z}_\gamma$. This in conjunction with (5.20)–(5.21) and the definition of S implies

$$\mathfrak{S}_3 \in C\big(\mathbb{B}_1(r_0,\delta), \mathbb{X}^s(\mathbb{R}_+,\delta)\big), \quad \mathfrak{S}_3(0) = 0. \tag{5.23}$$

Combining (5.20)–(5.23) then yields (5.19).

For given $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0,\delta)$ let $(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}) = \mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0, \bar{z})$. Then we have

$$\begin{aligned} x(t) &= -\int_t^\infty P^c\omega\bar{z}(\tau)\,d\tau = -\int_0^\infty P^c\omega\bar{z}(\tau)\,d\tau + \int_0^t P^c\omega\bar{z}(\tau)\,d\tau \\ &= \mathsf{x}_0 - \mathsf{x}_\infty + \int_0^t P^c\omega\bar{z}(\tau)\,d\tau, \end{aligned}$$

thus showing that x solves the first equation in (5.14). In summary, we have shown that $(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}) = \mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0, \bar{z})$ is for every $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0,\delta)$ the unique solution of (5.14) in $X_\gamma^c \times \mathbb{X}^c(\mathbb{R}_+,\delta) \times \mathbb{X}^s(\mathbb{R}_+,\delta)$, where $\delta \in [0,\delta_0]$.
Setting

$$\begin{aligned} \tilde{z} &= \tilde{\mathfrak{Z}}(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := \mathsf{x} + \psi(\mathsf{x}_\infty + \mathsf{x}) - \psi(\mathsf{x}_\infty) + \mathsf{y}, \\ z_\infty &= \mathfrak{Z}_\infty(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := \mathsf{x}_\infty + \psi(\mathsf{x}_\infty) + \phi(\mathsf{x}_\infty + \psi(\mathsf{x}_\infty)) \end{aligned} \tag{5.24}$$

for $(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}) = \mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0, \bar{z})$, we see that

$$\tilde{\mathfrak{Z}} \in C(\mathbb{B}_1(r_0, \delta), \mathbb{X}(\mathbb{R}_+, \delta)), \quad \tilde{\mathfrak{Z}}(0) = 0,$$

and

$$\mathfrak{Z}_\infty \in C(\mathbb{B}_1(r_0, \delta), Z_\infty), \quad \mathfrak{Z}_\infty(0) = 0, \tag{5.25}$$

where $Z_\infty = [W_p^2(\Omega\backslash\Gamma_*)\cap C(\bar{\Omega})] \times W_p^{4-1/p}(\Gamma_*)$. It then follows from the derivation of (5.13)–(5.14) that

$$(z_\infty, \tilde{z}) = (\mathfrak{Z}(\mathsf{x}_0, \mathsf{y}_0, \bar{z}), \tilde{\mathfrak{Z}}(\mathsf{x}_0, \mathsf{y}_0, \bar{z}))$$

is for every given $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0, \delta)$ the unique (global) solution of (5.10) with $\tilde{z}$ in the regularity class $\mathbb{X}(\mathbb{R}_+, \delta)$. In a next step we shall show that $\tilde{z}$ in fact has better regularity properties, namely

$$\tilde{\mathfrak{Z}} \in C(\mathbb{B}_1(r_0, \delta), \mathbb{E}(\mathbb{R}_+, \delta)), \quad \tilde{\mathfrak{Z}}(0) = 0. \tag{5.26}$$

In order to see this, let us first consider the case $\gamma \equiv 0$ (which implies $\gamma_* = 0$). From the fourth line of (5.10), the fact that $\tilde{z} \in \mathbb{X}(\mathbb{R}_+, \delta)$, and

$$[v \mapsto v|_{\Gamma_*}] \in \mathcal{B}(\mathbb{X}(\mathbb{R}_+, \delta), W_p^{1-1/2p}(\mathbb{R}_+, \delta; L_p(\Gamma_*)))$$

follows

$$\tilde{\rho} = (\mu - \sigma A_*)^{-1}((l_*/u_*)\tilde{v} + \mu\tilde{\rho}) \in W_p^{1-1/2p}(\mathbb{R}_+, \delta; H_p^2(\Gamma_*)),$$

where μ is in the resolvent set of σA_*. From the fifth line of (5.10), the fact that $(\tilde{z}, \bar{z}) \in \mathbb{X}(\mathbb{R}_+, \delta) \times \mathbb{E}(\mathbb{R}_+, \delta)$, and trace theory for $\tilde{v}$ follows

$$l_*\partial_t\tilde{\rho} = [\![d_*\partial_\nu\tilde{v}]\!] + l_*\omega\bar{\rho} \in W_p^{1/2-1/2p}(\mathbb{R}_+, \delta; L_p(\Gamma_*)),$$

implying that $\tilde{\rho} \in W_p^{3/2-1/2p}(\mathbb{R}_+, \delta; L_p(\Gamma_*))$. Hence (5.26) holds for $\gamma = 0$.

If $\gamma > 0$ (and thus $\gamma_* > 0$) we use the embedding

$$H_p^1(\mathbb{R}_+, \delta; W_p^{2-1/p}(\Gamma_*)) \cap L_p(\mathbb{R}_+, \delta; W_p^{4-1/p}(\Gamma_*)) \hookrightarrow W_p^{1-1/2p}(\mathbb{R}_+, \delta; H_p^2(\Gamma_*))$$

and the fourth equation in (5.10) to conclude that $\tilde{\rho} \in W_p^{2-1/2p}(\mathbb{R}_+, \delta; L_p(\Gamma_*))$. Hence (5.26) holds in this case as well.

Let us now turn our attention to equation (5.9), or equivalently, the first line of (5.11). In a similar way as in the proof of [25, Proposition 10] (extra consideration is needed in order to deal with the additional terms involving z_∞) one verifies that the mapping

$$[(z_\infty, z) \mapsto N(z_\infty + z) - N(z_\infty)] : \mathbb{U}(\delta) \to \mathbb{F}(\mathbb{R}_+, \delta)$$

is C^1 and vanishes together with its Fréchet derivative at $(0, 0)$. Here $\mathbb{U}(\delta)$ denotes an open neighborhood of $(0, 0)$ in $Z_\infty \times \mathbb{E}(\mathbb{R}_+, \delta)$. Let

$$\mathbb{B}(r, \delta) = \{(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in X_\gamma^c \times (X_\gamma^s \cap \tilde{Z}_\gamma) \times \mathbb{E}(\mathbb{R}_+, \delta) : |(\mathsf{x}_0, \mathsf{y}_0, \bar{z})|_{[\tilde{Z}_\gamma]^2 \times \mathbb{E}(\mathbb{R}_+, \delta)} < r_0\},$$

and let $\mathrm{ext}_\delta \in \mathcal{B}\big(W_p^{2-2/p}(\Omega \setminus \Gamma_*) \cap C(\overline{\Omega})) \times W_p^{4-3/p}(\Gamma_*), \mathbb{E}(\mathbb{R}_+, \delta)\big)$ be an appropriate extension operator with $(\mathrm{ext}_\delta w_0)(0) = w_0$.

For $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}(r_0, \delta)$, with r_0 sufficiently small, we define

$$M(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := N(z_\infty + \tilde{z} + \mathrm{ext}_\delta(\phi(\tilde{z}_0) - \phi(\tilde{z}_\infty) - \bar{z}(0)) + \bar{z}) - N(z_\infty).$$

It follows from (5.24)-(5.26) that $M \in C(\mathbb{B}(r_0,\delta), \mathbb{F}(\mathbb{R}_+,\delta))$, $M(0,0,0) = 0$, and $D_3M(0,0,0) = 0$. Moreover,

$$M(\mathsf{x}_0, \mathsf{y}_0, \bar{z})(0) = N(z_0) - N(z_\infty), \quad (\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}(r_0, \delta), \tag{5.27}$$

where we recall that $\tilde{z}(0) = \tilde{z}_0 - \tilde{z}_\infty$, $z_0 = \tilde{z}_0 + \phi(\tilde{z}_0)$, and $z_\infty = \tilde{z}_\infty + \phi(\tilde{z}_\infty)$.

Finally, for $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}(r_0, \delta)$ let

$$K(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := (\mathbb{L}_{\gamma,\omega}, \mathrm{tr})^{-1}(M(\mathsf{x}_0, \mathsf{y}_0, \bar{z}), \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty)). \tag{5.28}$$

It follows from (5.27) and the definition of ϕ that the functions

$$(M(\mathsf{x}_0, \mathsf{y}_0, \bar{z}), \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty))$$

satisfy the necessary compatibility conditions, whenever $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}(r_0, \delta)$. Slight modifications of the results in [9] then imply that $K : \mathbb{B}(r_0, \delta) \to \mathbb{E}(\mathbb{R}_+, \delta)$ is well-defined, provided ω is large enough (and δ is in $[0, \delta_0]$ with δ_0 as above). From the properties of the mappings N, ψ and ϕ, the definition of $\tilde{z}_0$ and $\tilde{z}_\infty$ (recall that $\tilde{z}_0 = \mathsf{x}_0 + \psi(\mathsf{x}_0) + \mathsf{y}_0$, $\tilde{z}_\infty = \mathsf{x}_\infty + \psi(\mathsf{x}_\infty)$), and the contraction mapping theorem follows that K, defined in (5.28), has for each $(\mathsf{x}_0, \mathsf{y}_0)$ sufficiently small a unique fixed point

$$\bar{z} = \bar{z}(\mathsf{x}_0, \mathsf{y}_0) \in \mathbb{E}(\mathbb{R}_+, \delta),$$

and that the mapping $[(\mathsf{x}_0, \mathsf{y}_0) \mapsto \bar{z}(\mathsf{x}_0, \mathsf{y}_0)]$ is continuous and vanishes at $(0, 0)$. By construction it follows that $\bar{z} = \bar{z}(\mathsf{x}_0, \mathsf{y}_0)$ solves

$$\mathbb{L}_{\gamma,\omega}\bar{z} = N(z_\infty + \tilde{z} + \bar{z}) - N(z_\infty), \quad \bar{z}(0) = \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty).$$

In summary, we have shown that for each $z_0 \in \mathcal{SM}_\gamma$ small enough, there exists

$$(z_\infty, \tilde{z}, \bar{z}) \in Z_\infty \times \mathbb{E}(\mathbb{R}_+, \delta) \times \mathbb{E}(\mathbb{R}_+, \delta)$$

such that $z = z_\infty + \tilde{z} + \bar{z}$ is the unique global solution of (5.1). In particular we have shown that for every $z_0 \in \mathcal{SM}_\gamma$ small enough there exists a unique equilibrium $z_\infty = z_\infty(z_0)$ such that the solution of (5.1) exists for all $t \geq 0$ and converges to z_∞ in $\mathcal{SM}_\gamma$ at an exponential rate.

(b) Now we consider the linearly unstable case and we first show that the equilibrium 0 is unstable for the nonlinear equation (5.1). Using the same notation as in part (a) we consider the system of equations

$$\begin{aligned} \mathbb{L}_{\gamma,\omega}\bar{z} &= N(\tilde{z} + \bar{z}), & \bar{z}(0) &= \phi(\tilde{z}_0), \\ \dot{\tilde{z}} + L_\gamma \tilde{z} &= \omega \bar{z}, & \tilde{z}(0) &= \tilde{z}_0. \end{aligned} \tag{5.29}$$

Given $\alpha \in \mathbb{R}$ one verifies (by similar considerations as in [25, Proposition 10]) that there is a nondecreasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\eta(r) \to 0$ as $r \to 0$ and

$$\|e^{\alpha t} N(z)\|_{\mathbb{F}(a)} \leq \eta(r) \|e^{\alpha t} z\|_{\mathbb{E}(a)}, \quad e^{\alpha t} z \in \mathbb{E}(a), \tag{5.30}$$

whenever $|z(t)|_{Z_\gamma} \le r$ for $0 \le t \le a$. Here $a > 0$ is an arbitrary fixed number. For later use we note that

$$\mathbb{E}(a) \hookrightarrow L_p([0,a]; X_\gamma), \tag{5.31}$$

where the embedding constant is independent of a.

Let σ^+ be the collection of all positive eigenvalues of $(-L_\gamma)$. and let P^+ be the spectral projection related to the spectral set σ^+. Additionally, let $P^- := I - P^+$ and $X_\gamma^\pm := P^\pm(X_\gamma)$. Then X_γ^+ is finite dimensional and we obtain the decomposition

$$X = X_\gamma^+ \oplus X_\gamma^-, \quad L_\gamma = L_\gamma^+ \oplus L_\gamma^-.$$

We note that $\sigma(-L_\gamma^+) = \sigma^+$ and $\sigma(-L_\gamma^-) \subset [\operatorname{Re} z \le 0]$, where $\sigma(-L_\gamma^\pm)$ denotes the spectrum of $(-L_\gamma^\pm)$, respectively. Let λ_* be the smallest positive eigenvalue of $(-L_\gamma^+)$ and choose positive numbers κ, μ such that $[\kappa - \mu, \kappa + \mu] \subset (0, \lambda_*)$. We remind that the spectrum of $(-L_\gamma)$ consists of real eigenvalues, so that the strip $[\kappa - \mu \le \operatorname{Re} z \le \kappa + \mu]$ does not contain any spectral values of $(-L_\gamma)$. Therefore, there exists a constant $M \ge 1$ such that

$$|e^{-L_\gamma^- t}| \le M e^{(\kappa-\mu)t}, \quad |e^{L_\gamma^+ t}| \le M e^{-(\kappa+\mu)t}, \quad t \ge 0. \tag{5.32}$$

Suppose now, by contradiction, that the equilibrium 0 is stable for (5.1). Then for every $r > 0$ there is a number $\delta > 0$ such that (5.1) admits a global solution $z \in \mathbb{E}(\mathbb{R}_+)$ with $|z(t)| \le r$ for all $t \ge 0$ whenever $z_0 \in \bar{B}_\delta(0)$.
In the following we will use the decomposition $z = \tilde{z} + \bar{z}$, where $(\tilde{z}, \bar{z})$ is the solution of the linear system (5.29). (The function $z = \tilde{z} + \bar{z}$ is known, so that the first equation has a unique solution $\bar{z}$. With $\bar{z}$ determined, $\tilde{z} = z - \bar{z}$ is the unique solution of the second equation.) The functions $P^\pm \tilde{z}$ satisfy

$$\frac{d}{dt} P^\pm \tilde{z} + L_\gamma^\pm P^\pm \tilde{z} = P^\pm \omega \bar{z}, \quad P^\pm \tilde{z}(0) = P^\pm \tilde{z}_0. \tag{5.33}$$

Next we shall show that $P^+ \tilde{z}$ is given by the formula

$$P^+ \tilde{z}(t) = -\int_t^\infty e^{-L_\gamma^+(t-\tau)} P^+ \omega \bar{z}\, d\tau, \quad t \ge 0. \tag{5.34}$$

Given any $a > 0$ it follows from $|P^+ \tilde{z}(t)|_{X_\gamma^+} \le r$ that

$$\|e^{-\kappa t} P^+ \tilde{z}\|_{L_p([0,a]; X_\gamma^+)} \le r \Big(\int_0^a e^{-\kappa p t}\, dt \Big)^{1/p} \le C(\kappa, p) r. \tag{5.35}$$

From the relation

$$\frac{d}{dt} e^{-\kappa t} P^+ \tilde{z} = (-\kappa - L_\gamma^+) e^{-\kappa t} P^+ \tilde{z} + e^{-\kappa t} P^+ \omega \bar{z}, \tag{5.36}$$

(5.35)-(5.36) and (5.31) follows

$$\|e^{-\kappa t} P^+ \tilde{z}\|_{\mathbb{X}(a)} \le C_1 \big(r + \|e^{-\kappa t} \bar{z}\|_{\mathbb{E}(a)} \big), \tag{5.37}$$

with a universal constant C_1. Here $\mathbb{X}(a)$ is defined as in (5.16), with the difference that $\mathbb{R}_+$ is replaced by the interval $[0, a]$ and $\delta = 0$. We also recall that X_γ^+ is

finite dimensional, so that the spaces X_γ^+ and $D(L_\gamma^+)$ coincide (and therefore carry equivalent norms). From semigroup theory, maximal regularity, (5.32)-(5.33) and (5.31) follows

$$\begin{aligned}\|e^{-\kappa t}P^-\tilde z\|_{\mathbb{X}(a)} &\le M\big(|P^-\tilde z_0| + \|e^{-\kappa t}P^-\omega\bar z\|_{L_p([0,a];X_\gamma)}\big)\\ &\le M\big(|P^-\tilde z_0| + C_2\|e^{-\kappa t}\bar z\|_{\mathbb{E}(a)}\big).\end{aligned}\tag{5.38}$$

Combining (5.37)-(5.38) results in

$$\|e^{-\kappa t}\tilde z\|_{\mathbb{X}(a)} \le C_3\big(r + |P^-\tilde z_0| + \|e^{-\kappa t}\bar z\|_{\mathbb{E}(a)}\big),\tag{5.39}$$

where C_3 is a universal constant. Similarly as in part (a) we can infer from the equation for $\tilde z$ that

$$\|e^{-\kappa t}\tilde z\|_{\mathbb{E}(a)} \le c(\|e^{-\kappa t}\tilde z\|_{\mathbb{X}(a)} + \|e^{-\kappa t}\bar z\|_{\mathbb{E}(a)}),\tag{5.40}$$

and this implies

$$\|e^{-\kappa t}\tilde z\|_{\mathbb{E}(a)} \le C_4\big(r + |P^-\tilde z_0| + \|e^{-\kappa t}\bar z\|_{\mathbb{E}(a)}\big)\tag{5.41}$$

with $C_4 = c(1+C_3)$. On the other hand we obtain from the equation for $\bar z$ and (5.30)

$$\begin{aligned}\|e^{-\kappa t}\bar z\|_{\mathbb{E}(a)} &\le \bar C\big(|\phi(\tilde z_0)| + \|e^{-\kappa t}N(\tilde z + \bar z)\|_{\mathbb{E}(a)}\big)\\ &\le \bar C\big(|\phi(\tilde z_0)| + \eta(r)(\|e^{-\kappa t}\tilde z\|_{\mathbb{E}(a)} + \|e^{-\kappa t}\bar z\|_{\mathbb{E}(a)})\big).\end{aligned}$$

If r is chosen small enough such that $\bar C\eta(r) \le 1/2$ then

$$\|e^{-\kappa t}\bar z\|_{\mathbb{E}(a)} \le 2\bar C\big(|\phi(\tilde z_0)| + \eta(r)\|e^{-\kappa t}\tilde z\|_{\mathbb{E}(a)}\big).\tag{5.42}$$

We can, at last, combine (5.41)–(5.42) to the result

$$\|e^{-\kappa t}\tilde z\|_{\mathbb{E}(a)} + \|e^{-\kappa t}\bar z\|_{\mathbb{E}(a)} \le C_5\big(r + |P^-\tilde z_0| + |\phi(\tilde z_0)|\big),\tag{5.43}$$

provided r is chosen small enough so that $2(1+C_4)\bar C\eta(r) \le 1/2$. Since all estimates are independent of a we conclude that $e^{-\kappa t}z \in \mathbb{E}(\mathbb{R}_+)$. From (5.43) and Hölder's inequality follows

$$\begin{aligned}&e^{-\kappa t}\int_t^\infty |e^{-L_\gamma^+(t-\tau)}P^+\omega\bar z(\tau)|_{X_\gamma^+}\,d\tau\\ &\le M\Big(\int_t^\infty e^{\mu p'(t-\tau)}\,d\tau\Big)^{1/p'}\|e^{-\kappa\tau}\omega\bar z\|_{L_p(\mathbb{R}_+:X_\gamma)} \le C\|e^{-\kappa t}\bar z\|_{\mathbb{E}(\mathbb{R}_+)} < \infty,\end{aligned}$$

thus showing that the integral $\int_t^\infty e^{-L_\gamma^+(t-\tau)}P^+\omega\bar z\,d\tau$ exists in X_γ^+ for every $t \ge 0$. Moreover, its norm in X_γ^+ grows no faster than the exponential function $e^{\kappa t}$. It follows from the variation of parameters formula that

$$e^{L_\gamma^+ t}\Big(P^+\tilde z(t) + \int_t^\infty e^{-L_\gamma^+(t-\tau)}P^+\omega\bar z(\tau)\,d\tau\Big) = P^+\tilde z_0 + \int_0^\infty e^{L_\gamma^+\tau}P^+\omega\bar z(\tau)\,d\tau,$$

and the estimate

$$\big|e^{L_\gamma^+ t}(P^+\tilde z(t) + \int_t^\infty e^{-L_\gamma^+(t-\tau)}P^+\omega\bar z\,d\tau)\big|_{X_\gamma^+} \le Me^{-(\kappa+\mu)t}(r + Ce^{\kappa t}), \quad t \ge 0,$$

then shows that $P^+\tilde z_0 + \int_0^\infty e^{L_\gamma^+\tau}P^+\omega\bar z\,d\tau = 0$. Thus the representation (5.34) holds as claimed. With this established, we obtain from Young's inequality for convolution integrals

$$\|e^{-\kappa t}P^+\tilde z(t)\|_{L_p(R_+,X_\gamma^+)} \le M\mu^{-1}\|e^{-\kappa t}P^+\omega\bar z\|_{L_p(\mathbb{R}_+;X_\gamma^+)}.$$

It then follows from (5.31) and (5.36) that

$$\|e^{-\kappa t}P^+\tilde z\|_{\mathbb{X}(\mathbb{R}_+)} \le C\|e^{-\kappa t}\bar z\|_{\mathbb{E}(\mathbb{R}_+)}. \tag{5.44}$$

We can now imitate the estimates in (5.38)-(5.42), with the interval $[0,a]$ replaced by $\mathbb{R}_+$, to conclude that

$$\|e^{-\kappa t}\tilde z\|_{\mathbb{E}(\mathbb{R}_+)} + \|e^{-\kappa t}\bar z\|_{\mathbb{E}(\mathbb{R}_+)} \le C_6\big(|P^-\tilde z_0| + |\phi(\tilde z_0)|\big). \tag{5.45}$$

This, in combination with (5.32), (5.34), and Hölder's inequality, yields the estimate

$$\begin{aligned}|P^+\tilde z_0|_{X_\gamma^+} &\le M\int_0^\infty e^{-\mu\tau}|e^{-\kappa\tau}P^+\omega\bar z|_{X_\gamma^+}\,d\tau\\ &\le C\|e^{-\kappa t}P^+\omega\bar z\|_{L_p(\mathbb{R}_+;X_\gamma^+)} \le C\big(|P^-\tilde z_0| + |\phi(\tilde z_0)|\big).\end{aligned}$$

By decreasing δ if necessary, we can assume that $C|\phi(\tilde z_0)| \le 1/2(|P^+\tilde z_0| + |P^-\tilde z_0|)$ for all $\tilde z_0 \in B_\delta(0)$. (Recall that $\phi(0) = \phi'(0) = 0$.) Hence

$$|P^+\tilde z_0|_{\tilde Z_\gamma} \le C_7|P^-\tilde z_0|_{\tilde Z_\gamma}, \quad \tilde z_0 \in B_\delta(0), \tag{5.46}$$

with a uniform constant C_7, and this shows that 0 cannot be stable for (5.1).

It remains to show the last assertion of Theorem 5.2(b). For this we consider the projection $P^u = I - P^c - P^s$ which projects onto X_γ^u, the unstable subspace of X_γ associated with the (finitely many) unstable eigenvalues. As in part (a) we will show that there exists an equilibrium z_∞ such that any solution that stays in a small neighborhood of 0 converges to $z_\infty = z_\infty(z_0)$ exponentially fast as $t \to \infty$. Using the decomposition $\mathsf{y} = \mathsf{y}_s + \mathsf{y}_u$, we obtain as in (a) the following system of equations:

$$\begin{cases} \dot{\mathsf{x}} = P^c\omega\bar z, & \mathsf{x}(0) = \mathsf{x}_0 - \mathsf{x}_\infty,\\ \dot{\mathsf{y}}_s + L_\gamma^s\,\mathsf{y}_s = S_s(\mathsf{x}_\infty,\mathsf{x},\bar z), & \mathsf{y}_s(0) = \mathsf{y}_0^s,\\ \dot{\mathsf{y}}_u + L_\gamma^u\,\mathsf{y}_u = S_u(\mathsf{x}_\infty,\mathsf{x},\bar z), & \mathsf{y}_u(0) = \mathsf{y}_0^u,\end{cases} \tag{5.47}$$

with

$$S_j(\mathsf{x}_\infty,\mathsf{x},\bar z) = P^j\omega\bar z - \psi_j'(\mathsf{x}_\infty + \mathsf{x})P^c\omega\bar z - L_\gamma^j[\psi_j(\mathsf{x}_\infty + \mathsf{x}) - \psi_j(\mathsf{x}_\infty)],$$

where $j \in \{s,u\}$, and where the functions ψ_j are defined similarly as in (5.12). Suppose we have a global solution $z \in \mathbb{E}(\mathbb{R}_+)$ of (5.1) with $z(0) = z_0 \in \mathcal{SM}_\gamma$ which satisfies $|z|_{\tilde Z_\gamma} \le r$, where $r > 0$ is sufficiently small. By similar arguments as above (the presence of the function S_u does not cause any principal difficulties) we infer that

$$\mathsf{y}_u(t) = -\int_t^\infty e^{-L_\gamma^u(t-\tau)}S_u(\mathsf{x}_\infty,\mathsf{x},\bar z)\,d\tau, \quad t \ge 0. \tag{5.48}$$

For $(\mathsf{x}_0, \mathsf{y}_0^s, \bar z) \in \mathbb{B}_1(r_0, \delta)$, with r_0 sufficiently small, we set

$$\begin{aligned} \mathsf{x}_\infty &:= \mathsf{x}_0 + \int_0^\infty P^c \omega \bar z(\tau)\, d\tau, \\ \mathsf{x}(t) &:= -\int_t^\infty P^c \omega \bar z(\tau)\, d\tau, \\ \mathsf{y}_s &:= \Big(\frac{d}{dt} + L_\gamma^s, \mathrm{tr}\Big)^{-1} \big(S_s(\mathsf{x}_\infty, \mathsf{x}, \bar z), \mathsf{y}_0^s\big), \\ \mathsf{y}_u(t) &:= -\int_t^\infty e^{-L_\gamma^u (t-\tau)} S_u(\mathsf{x}_\infty, \mathsf{x}, \bar z)\, d\tau. \end{aligned} \tag{5.49}$$

As in part (a) we conclude that (5.49) admits for each $(\mathsf{x}_0, \mathsf{y}_0^s, \bar z) \in \mathbb{B}_1(r_0, \delta)$, with r_0 sufficiently small, a unique solution

$$(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}_s, \mathsf{y}_u) = \mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0^s, \bar z) \in X_\gamma^c \times \mathbb{X}^c(\mathbb{R}_+, \delta) \times \mathbb{X}^s(\mathbb{R}_+, \delta) \times \mathbb{X}^u(\mathbb{R}_+, \delta).$$

Following the arguments of part (a) then renders a solution

$$\mathfrak{Z}(\mathsf{x}_0, \mathsf{y}_0^s) = z_\infty + \mathsf{x} + \psi(\mathsf{x} + \mathsf{x}_\infty) - \psi(\mathsf{x}_\infty) + \mathsf{y}_s + \mathsf{y}_u + \bar z$$

of (5.1) with $z_0 = \mathsf{x}_0 + \psi(\mathsf{x}_0) + \mathsf{y}_0^u + \mathsf{y}_0^s + \phi(\mathsf{x}_0 + \psi(\mathsf{x}_0) + \mathsf{y}_0^u + \mathsf{y}_0^s)$, where y_0^u is determined by

$$\mathsf{y}_0^u = -\int_0^\infty e^{L_\gamma^u \tau} S_u(\mathsf{x}_\infty, \mathsf{x}, \bar z)\, d\tau. \tag{5.50}$$

The solution $\mathfrak{Z}(\mathsf{x}_0, \mathsf{y}_0^s)$ converges exponentially fast toward the equilibrium z_∞. In addition, we have shown that the initial value z_0 necessarily lies on the stable manifold belonging to z_∞, determined by the relation (5.50).

Due to uniqueness of (local) solutions to (5.1), the solution $\mathfrak{Z}(\mathsf{x}_0, \mathsf{y}_0^s)$ coincides with the given global solution z, and the proof of part (b) is now complete. ☐

Global existence and convergence. There are several obstructions against global existence for the Stefan problem (1.1):

- *regularity*: the norms of either $u(t)$, $\Gamma(t)$, and in addition $[\![d\partial_\nu u(t)]\!]$ in case $\gamma \equiv 0$, become unbounded;
- *well-posedness*: in case $\gamma \equiv 0$ the well-posedness condition $l(u) \neq 0$ may become violated; or u may become 0;
- *geometry*: the topology of the interface changes;
  or the interface touches the boundary of Ω;
  or the interface contracts to a point.

Note that the compatibility conditions $[\![\psi(u)]\!] + \sigma\mathcal{H} = 0$ in case $\gamma \equiv 0$, and

$$(l(u) - [\![\psi(u)]\!] - \sigma\mathcal{H})([\![\psi(u)]\!] + \sigma\mathcal{H}) = \gamma(u)[\![d\partial_\nu u]\!]$$

in case $\gamma > 0$ are preserved by the semiflow.

Let (u,Γ) be a solution in the state manifold $\mathcal{SM}_\gamma$. By a *uniform ball condition* we mean the existence of a radius $r_0>0$ such that for each t, at each point $x\in\Gamma(t)$ there exist centers $x_i\in\Omega_i(t)$ such that $B_{r_0}(x_i)\subset\Omega_i$ and $\Gamma(t)\cap\bar{B}_{r_0}(x_i)=\{x\}$, $i=1,2$. Note that this condition bounds the curvature of $\Gamma(t)$, prevents it from shrinking to a point, from touching the outer boundary $\partial\Omega$, and from undergoing topological changes.

With this property, combining the semiflow for (1.1) with the Lyapunov functional and compactness we obtain the following result.

Theorem 5.3. *Let $p>n+2$, $\sigma>0$, suppose $\psi,\gamma\in C^3(0,\infty)$, $d\in C^2(0,\infty)$ such that either $\gamma\equiv 0$ or $\gamma(u)>0$ on $(0,\infty)$, and assume*

$$\kappa(u)=-u\psi''(u)>0,\quad d(u)>0,\quad u\in(0,\infty).$$

Suppose that (u,Γ) is a solution of (1.1) in the state manifold $\mathcal{SM}_\gamma$ on its maximal time interval $[0,t_*)$. Assume the following on $[0,t_*)$:

(i) $|u(t)|_{W_p^{2-2/p}}+|\Gamma(t)|_{W_p^{4-3/p}}\le M<\infty$;

(ii) $|[\![d\partial_\nu u(t)]\!]|_{W_p^{2-6/p}}\le M<\infty$ *in case* $\gamma\equiv 0$;

(iii) $|l(u(t))|\ge 1/M$ *in case* $\gamma\equiv 0$;

(iv) $u(t)\ge 1/M$;

(v) $\Gamma(t)$ *satisfies a uniform ball condition.*

Then $t_*=\infty$, i.e. the solution exists globally. If its limit set contains a stable equilibrium $(u_\infty,\Gamma_\infty)\in\mathcal{E}$, i.e. $\varphi'(u_\infty)<0$, then it converges in $\mathcal{SM}_\gamma$ to this equilibrium. On the contrary, if $(u(t),\Gamma(t))$ is a global solution in $\mathcal{SM}_\gamma$ which converges to an equilibrium (u_*,Γ_*) with $l(u_*)\neq 0$ in case $\gamma\equiv 0$ in $\mathcal{SM}_\gamma$ as $t\to\infty$, then the properties (i)-(v) are valid.

Proof. Assume that assertions (i)–(v) are valid. Then $\Gamma([0,t_*))\subset W_p^{4-3/p}(\Omega,r)$ is bounded, hence relatively compact in $W_p^{4-3/p-\varepsilon}(\Omega,r)$. (See (3.7) for the definition of $W_p^s(\Omega,r)$.)

Thus we may cover this set by finitely many balls with centers Σ_k real analytic in such a way that $\mathrm{dist}_{W_p^{4-3/p-\varepsilon}}(\Gamma(t),\Sigma_j)\le\delta$ for some $j=j(t)$, $t\in[0,t_*)$. Let $J_k=\{t\in[0,t_*):\, j(t)=k\}$. Using for each k a Hanzawa-transformation Ξ_k, we see that the pull backs $\{u(t,\cdot)\circ\Xi_k:\, t\in J_k\}$ are bounded in $W_p^{2-2/p}(\Omega\setminus\Sigma_k)$, hence relatively compact in $W_p^{2-2/p-\varepsilon}(\Omega\setminus\Sigma_k)$. Employing now Corollary 3.10 we obtain solutions (u^1,Γ^1) with initial configurations $(u(t),\Gamma(t))$ in the state manifold on a common time interval, say $(0,\tau]$, and by uniqueness we have

$$(u^1(\tau),\Gamma^1(\tau))=(u(t+\tau),\Gamma(t+\tau)).$$

Continuous dependence implies then relative compactness of $(u(\cdot),\Gamma(\cdot))$ in $\mathcal{SM}_\gamma$. In particular, $t_*=\infty$ and the orbit $(u,\Gamma)(\mathbb{R}_+)\subset\mathcal{SM}_\gamma$ is relatively compact. The negative total entropy is a strict Lyapunov functional, hence the limit set $\omega(u,\Gamma)\subset\mathcal{SM}_\gamma$ of a solution is contained in the set $\mathcal{E}$ of equilibria. By compactness

$\omega(u,\Gamma) \subset \mathcal{SM}_\gamma$ is non-empty, hence the solution comes close to $\mathcal{E}$, and stays there. Then we may apply the convergence result Theorem 5.2. The converse is proved by a compactness argument. □

Remark 5.4. We believe that the extra assumption $\varphi'(u_\infty) < 0$ in Theorem 5.3 can be replaced by $\varphi'(u_\infty) \neq 0$. However, to prove this requires more technical efforts, and we refrain from doing this here.

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